Colloquium du CERMICS

École des Ponts ParisTech

Randomness in Partial Differential Equations

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Colloquium Cermics

Randomness in Partial Differential Equations

Max-Planck-Institut für **Mathematik in den Naturwissenschaften, Leipzig**

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Effective behavior of random media

=Stochastic homogenization:

Early explicit asymptotic treatment,

recent numerical applications

Maxwell: Effective resistance of a composite

A TREATISE

ON

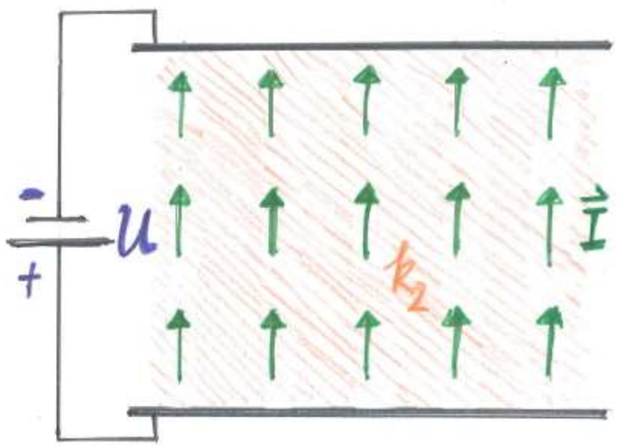
ELECTRICITY AND MAGNETISM

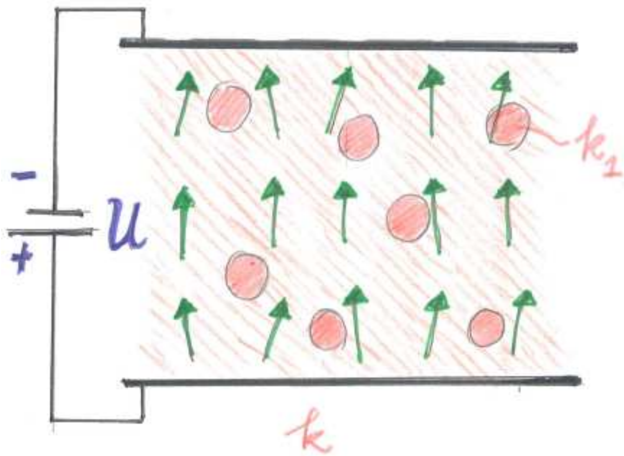
MAXWELL

That the one expression should be equivalent to the other,

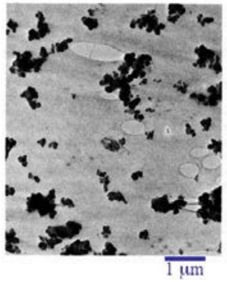
$$K = \frac{2k_1 + k_2 + p(k_1 - k_2)}{2k_1 + k_2 - 2p(k_1 - k_2)} k_2. \tag{17}$$

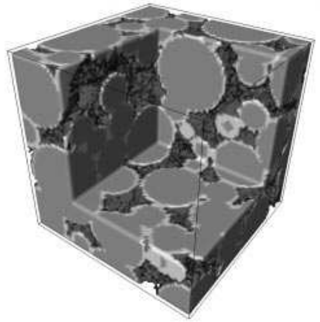
This, therefore, is the specific resistance of a compound medium consisting of a substance of specific resistance k_2, in which are disseminated small spheres of specific resistance k_1, the ratio of the volume of all the small spheres to that of the whole being p. In order that the action of these spheres may not produce effects depending on their interference, their radii must be small compared with their distances, and therefore p must be a small fraction.





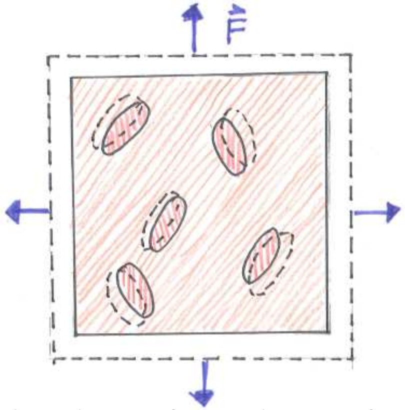
Recent: composite materials & porous media





Effective elasticity

Effective permeability



Effective behavior by simulation
of "Representative Volume Element"
Mathematical theory on qualitative level:
Varadhan&Papanicolaou, Kozlov '79, H-convergence by Murat&Tartar

Random medium ...

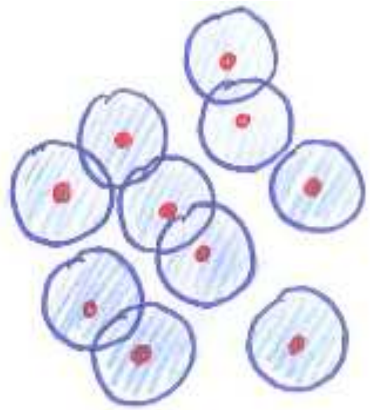
symmetric coefficient field $a = a(x)$ on d-dimensional space

$\lambda|\xi|^2 \leq \xi \cdot a(x)\xi \leq |\xi|^2$ for all points x and vectors ξ

$\rightsquigarrow$ uniformly elliptic operator $-\nabla \cdot a\nabla u$

Ensemble $\langle\cdot\rangle$ of such coefficient fields a

Example of ensemble $\langle\cdot\rangle$:
points Poisson distributed with density 1,
union of balls of radius $\frac{1}{4}$ around points,
$a =$ id on union, $a = \lambda$id on complement,



Stationarity: a and $a(y + \cdot)$ have same distribution under $\langle\cdot\rangle$

... = elliptic operator with random stationary coefficient field

Plan for talk

1) Error in Representative Volume Element (RVE) Method:
Scaling of random and systematic contribution
in terms of RVE-size

2) Fluctuations of macroscopic observables:
leading-order pathwise characterization,
RVE method for extraction

3) Stochastic homogenization $\leftrightarrow$ stochastic PDE:
"quenched noise" vs. "thermal noise",
many analogies

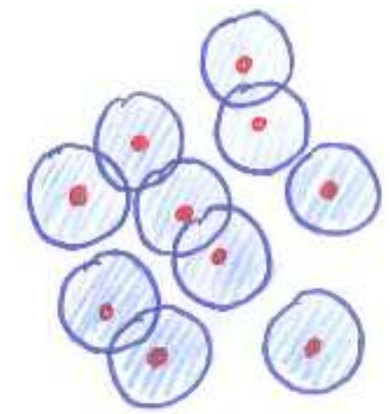
Representative Volume Element method

to extract effective tensor $\bar{a}$:

Scaling of random and systematic error in RVE size

A. Gloria, S. Neukamm

Goal: Extract effective behavior $\bar{a}$ from $\langle\cdot\rangle$ ...



Recall example of ensemble $\langle\cdot\rangle$:
points Poisson distributed with density 1,
union of balls of radius $\frac{1}{4}$ around points,
$a = \mathrm{id}$ on union, $a = \lambda\,\mathrm{id}$ on complement,

ensemble $\langle\cdot\rangle$ $\rightsquigarrow$ effective conductivity $\bar{a}$

$$\left\{\begin{array}{c}\text{density of points } 1\\ \text{radius of inclusions } \frac{1}{4}\\ \text{conductivity in pores } \lambda\end{array}\right\} \rightsquigarrow \bar{a} = \begin{pmatrix}\bar{a}_{11} & \bar{a}_{12}\\ \bar{a}_{21} & \bar{a}_{22}\end{pmatrix} = \bar{\lambda}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

3 numbers $\rightsquigarrow$ 1 number

... via Representative Volume Element (RVE)

Representative Volume Element method

Introduce artificial period L

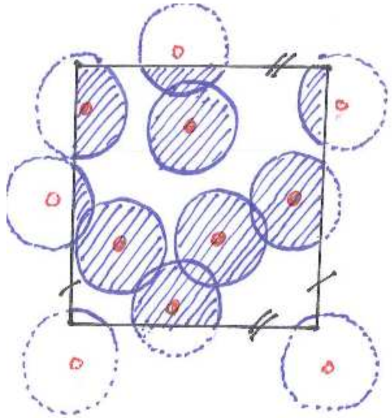
Periodized ensemble $\langle\cdot\rangle_L$

points Poisson distributed with density 1,

on d-dimensional torus $[0,L)^d$

union of balls of radius $\frac{1}{4}$ around points,

$a =$ id on union, $a = \lambda$ id on complement,



Given coordinate direction $i = 1, \cdots, d$ seek L-periodic φ_i with

$$-\nabla \cdot a(e_i + \nabla\varphi_i) = 0 \quad \text{on } [0,L)^d.$$

Spatial average $\fint_{[0,L)^d} a(e_i + \nabla\varphi_i)$ of flux $a(e_i + \nabla\varphi_i)$

as approximation to $\bar{a}e_i$ for $L \gg 1$;

φ_i is approximate "corrector", e_i unit vector in i-th coordinate direction

Solving d elliptic equations $-\nabla \cdot a(e_i + \nabla\varphi_i) = 0$ ...

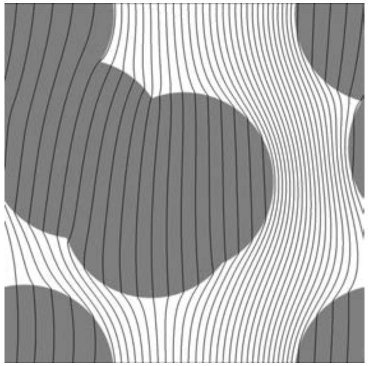
direction e_1

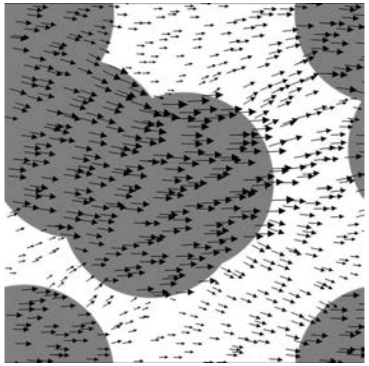
potential

$x_1 + \varphi_1$

flux

$a(e_1 + \nabla\varphi_1)$





average flux

$\fint a(e_1 + \nabla\varphi_1)$

$= \begin{pmatrix} 0.49641 \\ -0.02137 \end{pmatrix}$

$\approx \bar{a}e_1$

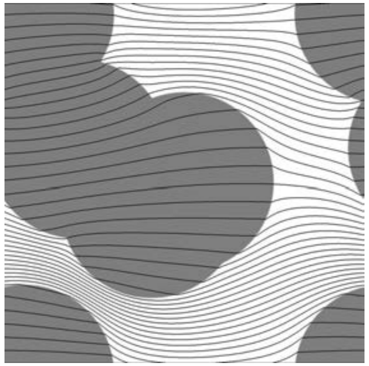
direction e_2

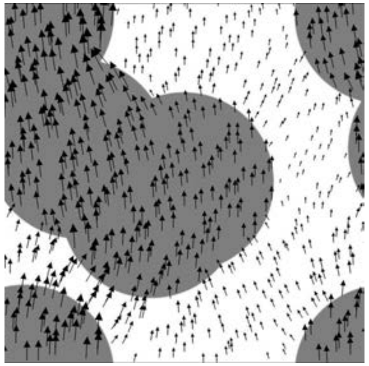
potential

$x_2 + \varphi_2$

flux

$a(e_2 + \nabla\varphi_1)$





average flux

$\fint a(e_2 + \nabla\varphi_2)$

$= \begin{pmatrix} -0.02137 \\ 0.53240 \end{pmatrix}$

$\approx \bar{a}e_2$

simulations by R. Kriemann (MPI)

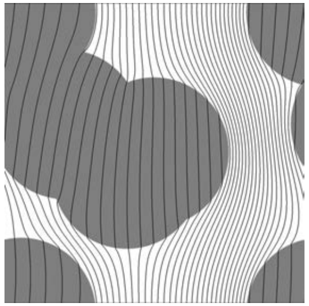
... gives approximation $\bar{a}_L$

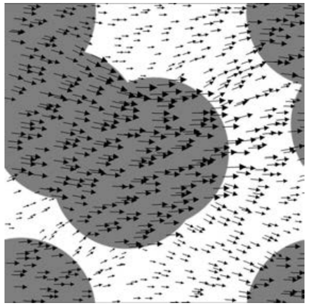
Random error: approx. $\bar{a}_L$ depends on realization

realization 1

potential,

current



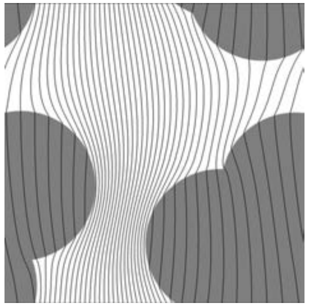


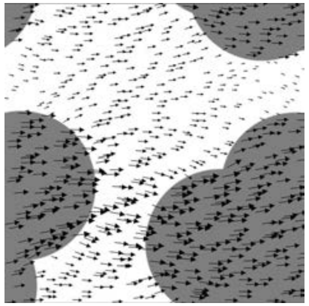
$$\bar{a}_L = \begin{pmatrix} 0.49641 & -0.02137 \\ -0.02137 & 0.53240 \end{pmatrix}$$

realization 2

potential,

current



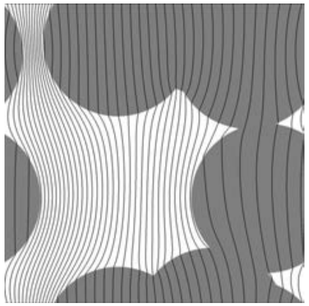


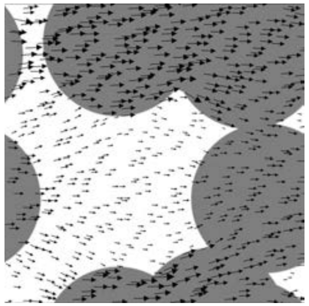
$$\bar{a}_L = \begin{pmatrix} 0.45101 & 0.01104 \\ 0.01104 & 0.45682 \end{pmatrix}$$

realization 3

potential,

current

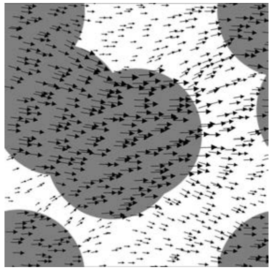




$$\bar{a}_L = \begin{pmatrix} 0.56213 & 0.00857 \\ 0.00857 & 0.60043 \end{pmatrix}$$

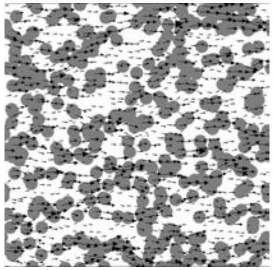
... and thus fluctuates / is random

Fluctuations of $\bar{a}_L$ decrease with increasing L



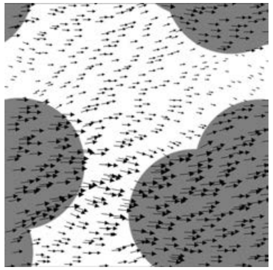
$\bar{a}_L =$

$$\begin{pmatrix} 0.50 & -0.02 \\ -0.02 & 0.53 \end{pmatrix}$$



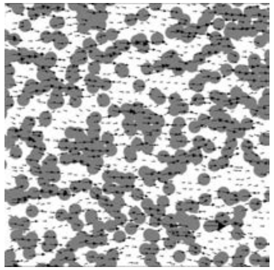
$\bar{a}_L =$

$$\begin{pmatrix} 0.518 & 0.004 \\ 0.004 & 0.511 \end{pmatrix}$$



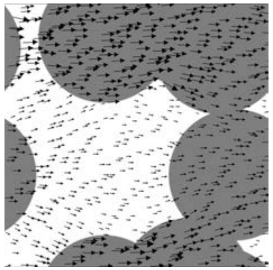
$\bar{a}_L =$

$$\begin{pmatrix} 0.45 & 0.01 \\ 0.01 & 0.46 \end{pmatrix}$$



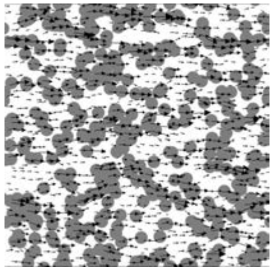
$\bar{a}_L =$

$$\begin{pmatrix} 0.532 & 0.005 \\ 0.005 & 0.523 \end{pmatrix}$$



$\bar{a}_L =$

$$\begin{pmatrix} 0.56 & 0.01 \\ 0.01 & 0.60 \end{pmatrix}$$



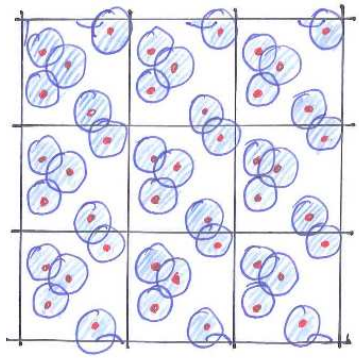
$\bar{a}_L =$

$$\begin{pmatrix} 0.515 & -0.001 \\ -0.001 & 0.521 \end{pmatrix}$$

... scaling of variance $\mathrm{var}(\bar{a}_L)$ **in** L**?**

Systematic error, decreases with increasing L

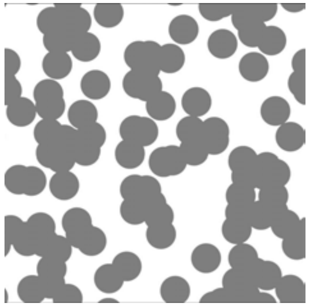
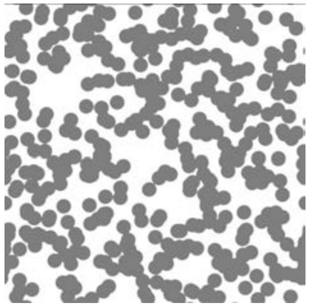
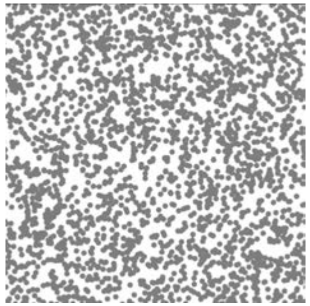
Also expectation $\langle \bar{a}_L \rangle_L$ depends on L
since from $\langle \cdot \rangle$ to $\langle \cdot \rangle_L$ statistics are altered
by artificial long-range correlations



$$\langle \bar{a}_L \rangle_L = \bar{\lambda}_L \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$ because of symmetry of $\langle \cdot \rangle$ under rotation

$L = 2$ $L = 5$ $L = 10$ $L = 20$ $L = 50$



0.551 0.524 0.520 0.522 0.522

Scaling of both errors in L ...

Pick a according to $\langle\cdot\rangle_L$, **solve** for φ (period L),

compute spatial average $\bar{a}_L e_i := \fint_{[0,L)^d} a(e_i + \nabla\varphi_i)$

Take *random variable* $\bar{a}_L$ **as approximation to** $\bar{a}$

$\langle$**error**$^2\rangle_L$ = **random**^{2} + **systematic**^{2}:

$$\langle|\bar{a}_L - \bar{a}|^2\rangle_L = \mathrm{var}_{\langle\cdot\rangle_L}[\bar{a}_L] + |\langle\bar{a}_L\rangle_L - \bar{a}|^2$$

Qualitative theory yields:

$$\lim_{L\uparrow\infty} \mathrm{var}_{\langle\cdot\rangle_L}[\bar{a}_L] = 0, \qquad \lim_{L\uparrow\infty} \langle\bar{a}_L\rangle_L = \bar{a}$$

... why rate is of interest?

Number of samples N vs. artificial period L

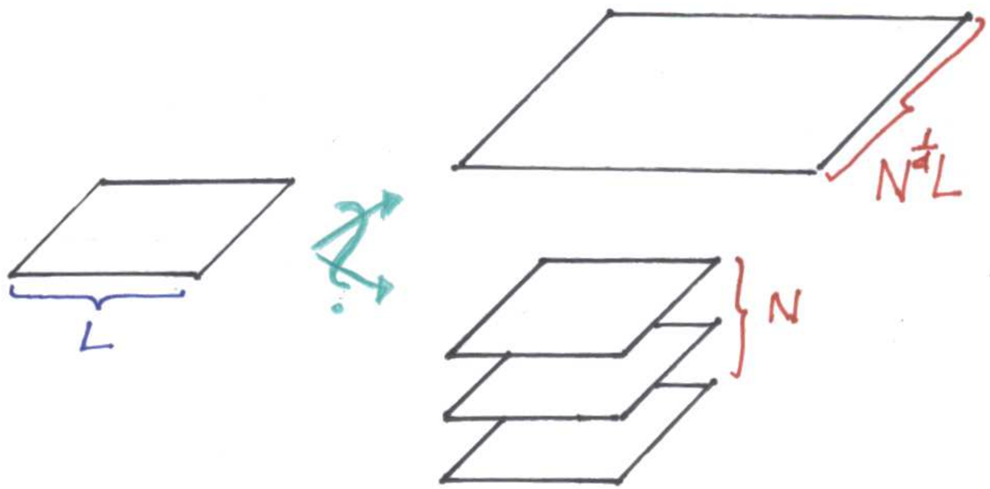
Take **N** samples, i. e. independent picks $a^{(1)}, \cdots, a^{(N)}$ from $\langle\cdot\rangle_L$.

Compute empirical mean $\frac{1}{N}\sum_{n=1}^{N} \fint_{[0,L)^d} a^{(n)}(e_i + \nabla\varphi_i^{(n)})$

$$\langle \textbf{total error}^2\rangle_L = \frac{1}{\mathbf{N}}\,\textbf{random error}^2 + \textbf{systematic error}^2$$

L ↑ reduces
systematic error and
random error

N ↑ reduces only
effect of **random error**



An optimal result

Let $\langle\cdot\rangle_L$ be ensemble of a's with period L,
with $\langle\cdot\rangle_L$ suitably coupled to $\langle\cdot\rangle$

For a with period L
solve $\nabla\cdot a(e_i+\nabla\varphi_i)=0$ for φ_i of period L.
Set $\bar a_L e_i = \fint_{[0,L)^d} a(e_i+\nabla\varphi_i)$.

Theorem [Gloria&O.'13, G.&Neukamm&O. Inventiones'15]

Random error$^2 = \mathrm{var}_{\langle\cdot\rangle_L}[\bar a_L] \;\le\; C(d,\lambda)L^{-d}$

Systematic error$^2 = \left|\langle\bar a_L\rangle_L - \bar a\right|^2 \;\le\; C(d,\lambda)L^{-2d}\ln^d L$

Gloria&Nolen '14: (random) error approximately Gaussian

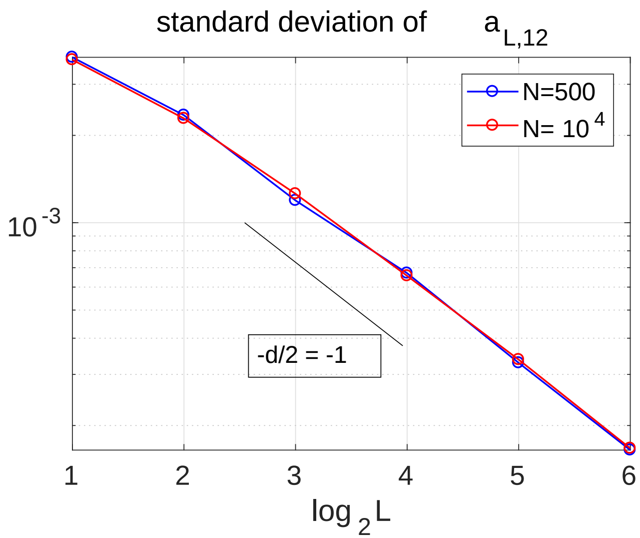
Fischer '17: variance reduction

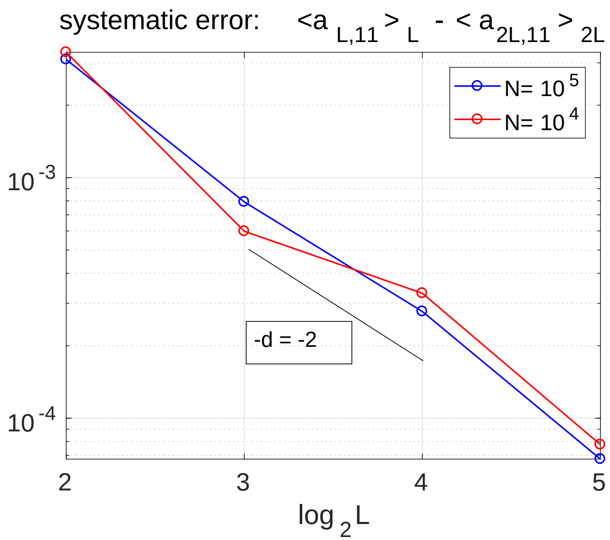
Numerical experiments display optimality

Random error $= \mathrm{var}^{\frac{1}{2}}_{\langle\cdot\rangle_L}[\bar{a}_L] \ \leq \ C(d,\lambda) L^{-\frac{d}{2}}$

Systematic error $= \left|\langle\bar{a}_L\rangle_L - \bar{a}\right| \ \leq \ C(d,\lambda) L^{-d} \ln^{\frac{d}{2}} L$

$\lim_{N\uparrow\infty}\left(\frac{1}{N}\sum_{n=1}^{N}(\bar{a}_{L,12}^{(n)})^2\right)^{\frac{1}{2}}$ $\qquad$ $|\lim_{N\uparrow\infty}\left(\frac{1}{N}\sum_{n=1}^{N}\bar{a}_{L,11}^{(n)} - \frac{1}{N}\sum_{n=1}^{N}\bar{a}_{2L,11}^{(n)}\right)|$





simulations by V. Khoromskaja (MPI) for $d=2$, different ensemble

State of art in quantitative stochastic homogenization ...

Yurinskii '86 : suboptimal rates in L for mixing $\langle\cdot\rangle$

Naddaf & Spencer '98, & Conlon '00:
optimal rates for small contrast $1-\lambda \ll 1$,
for $\langle\cdot\rangle$ with spectral gap

Gloria & O. '11, & Neukamm '13, & Marahrens '13:
optimal rates for all $\lambda > 0$ for $\langle\cdot\rangle$ with spectral gap,
Logarithmic Sobolev (concentration of measure)

Armstrong & Smart '14, & Mourrat '14, & Kuusi '15,
Gloria & O. '15
optimal stochastic integrability for finite range $\langle\cdot\rangle$

... of linear equations in divergence form

Homogenization error on macroscopic observables

Characterization of leading-order variances via a pathwise characterization of leading-order fluctuations

M. Duerinckx, A. Gloria

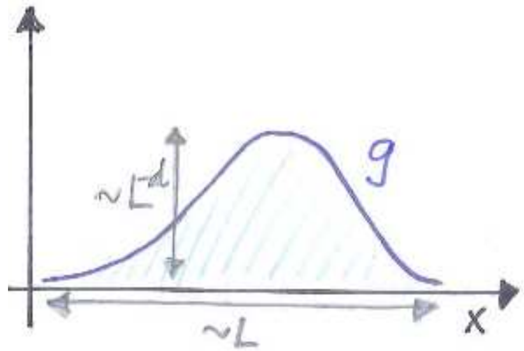
Macroscopic r. h. s. and observable ...

solution u of $\nabla\cdot a\nabla u=\nabla\cdot f$,

where r. h. s. $f(x)=\hat f(\frac{x}{L})$ deterministic

macroscopic observable $\int g\cdot\nabla u$,

where $g(x)=L^{-d}\hat g(\frac{x}{L})$ deterministic



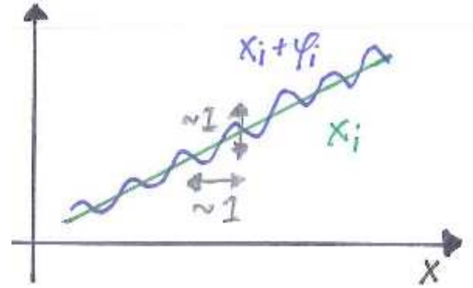
Marahrens & O.'13: $\mathrm{var}(\int g\cdot\nabla u)=O(\frac{1}{L^d})$

Goal: Characterize limiting variance $\lim_{L\uparrow\infty}L^d\mathrm{var}(\int g\cdot\nabla u)$

Naive approach via two-scale expansion

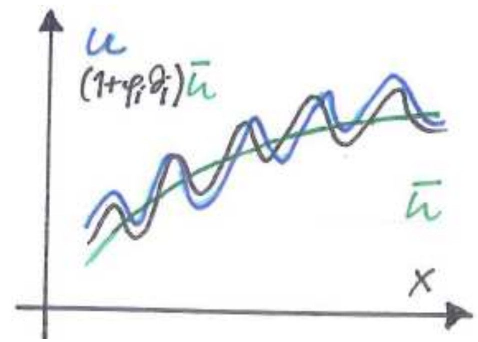
Goal: Characterize limiting variance $\lim_{L\uparrow\infty} L^d \mathrm{var}(\int g\cdot\nabla u)$

Corrector φ_i corrects affine x_i
such that $-\nabla\cdot a(e_i+\nabla\varphi_i)=0$
for coordinate direction $i=1,\cdots,d$



Solution $\bar u$ of homogenized equation
$\nabla\cdot(\bar a\nabla\bar u+f)=0$

Compare u to "two-scale expansion"
$(1+\varphi_i\partial_i)\bar u$ Einstein's summation rule



Naively expect $\mathrm{var}(\int g\cdot\nabla u) = \mathrm{var}(\int\nabla\cdot g\, u) \approx \mathrm{var}(\int\nabla\cdot g\,(1+\varphi_i\partial_i)\bar u)$

Hence study asymptotic covariance $\langle\varphi_i(x-y)\varphi_j(0)\rangle$

The subtle role of the two-scale expansion

Mourrat&O.'14: $\lim_{L\uparrow\infty} L^{d-2}\langle\varphi_i(L(\hat{x}-\hat{y}))\varphi_j(0)\rangle$ exists,
but $\neq$ a Green function $\bar{G}(\hat{x}-\hat{y})$ (Gaussian free field)
Helffer-Sjöstrand, annealed Green's function bounds $\rightsquigarrow$ 4-tensor $\bar{Q}$

Gu&Mourrat'15: $\lim_{L\uparrow\infty} L^{d}\mathrm{var}(\int g\cdot\nabla u)$ exists,
but $\neq$ $\lim_{L\uparrow\infty} L^{d}\mathrm{var}(\int \nabla\cdot g\,(1+\varphi_i\partial_i)\bar{u})$
Helffer-Sjöstrand $\rightsquigarrow$ same 4-tensor $\bar{Q}$, Gaussianity, heuristics
i. e. two-scale expansion cannot be applied naively

Duerinckx&Gloria&O.'16: Two-scale expansion
$\nabla u\approx\partial_i\bar{u}(e_i+\nabla\varphi_i)$ ok on level of "commutator"

$$\underbrace{a\nabla u}_{\text{flux}}-\bar{a}\underbrace{\nabla u}_{\text{field}}\approx\partial_i\bar{u}\Big(\underbrace{a(e_i+\nabla\varphi_i)-\bar{a}(e_i+\nabla\varphi_i)}_{=:\,\Xi_i}\Big)$$

Leading-order fluctuation of macro observables ...

$\Xi e_i = a(e_i+\nabla\varphi_i) - \bar{a}(e_i+\nabla\varphi_i)$ stationary tensor field

I) fluctuations commutator $\rightsquigarrow$ fluctuations observable

$\int g\cdot\nabla u = \int \nabla\bar{v}\cdot(a\nabla u - \bar{a}\nabla u) +$ determ.,

where $\bar{v}$ solves dual equation $\nabla\cdot(\bar{a}^*\nabla\bar{v} + g) = 0$.

II) $a\nabla u$-$\bar{a}\nabla u \approx \Xi\nabla\bar{u}$ holds in quantitative sense of

$L^d\mathrm{var}\big(\int g\cdot(a\nabla u\text{-}\bar{a}\nabla u - \Xi\nabla\bar{u})\big) = O(L^{-2})$.

III) Ξ is local, ie $\Xi(a,x)$ depends little on $a(y)$ for $|y-x|\gg 1$,

thus $\Xi \approx$ tensorial white noise on large scales

more precisely, $L^d\big|\mathrm{var}(\int g\cdot\Xi f) - \int f\otimes g : \bar{Q} f\otimes g\big| = O(L^{-2})$

for four-tensor $\bar{Q}$ from Mourrat&O.

I-III) $L^d\big|\mathrm{var}(\int g\cdot\nabla u) - \int \nabla\bar{v}\otimes\nabla\bar{u} : \bar{Q}\nabla\bar{v}\otimes\nabla\bar{u}\big| = O(L^{-2})$

... characterized via homogenization commutator

How to extract $\bar{Q}$ from $\langle\cdot\rangle$?

Recall standard commutator $\Xi e_i = a(e_i+\nabla\varphi_i)-\bar{a}(e_i+\nabla\varphi_i)$

$$L^d\mathrm{var}\Big(\int g\cdot\nabla u-\int\nabla\bar{v}\cdot\Xi\nabla\bar{u}\Big) = O(L^{-2}), \qquad \nabla\cdot(\bar{a}^*\nabla\bar{v}+g)=0$$

$$L^d\Big|\mathrm{var}\big(\int g\cdot\Xi f\big)-\int f\otimes g:\bar{Q}f\otimes g\Big| = O(L^{-2})$$

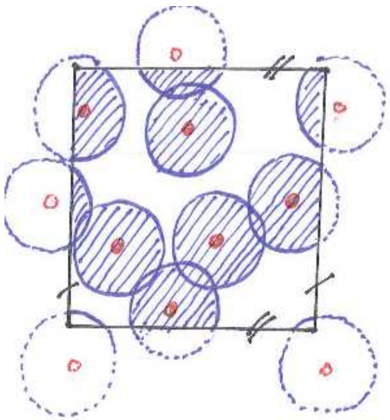
Duerinckx&Gloria&O.'17:

$$|L^d\mathrm{var}_{\langle\cdot\rangle_L}(\bar{a}_L)-\bar{Q}|\le C(d,\lambda)L^{-d}\ln^d L\ ,$$

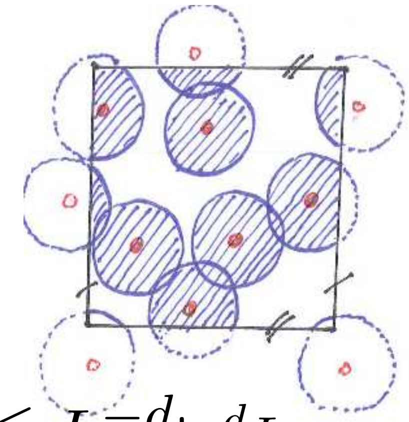
recall: $\langle\cdot\rangle_L$ ensemble of a's with period L,

solve $\nabla\cdot a(e_i+\nabla\varphi_i)=0$ for φ_i of period L,

Set $\bar{a}_L e_i = ⨍_{[0,L)^d} a(e_i+\nabla\varphi_i)$.



In practise: Extract $\bar{Q}$ from RVE ...



Recall periodized ensemble $\langle\cdot\rangle_L$

$\bar{a}_L e_i = \fint_{[0,L)^d} a(e_i + \nabla\varphi_i)$

Previous result: $|\langle\bar{a}_L\rangle_L - \bar{a}|^2 \lesssim L^{-2d}\ln^d L$

Duerinckx&Gloria&O.'17: $|L^d \mathrm{var}_{\langle\cdot\rangle_L}(\bar{a}_L) - \bar{Q}|^2 \lesssim L^{-d}\ln^d L$

Hence get $\bar{a}$ and $\bar{Q}$ by same procedure:

$N \sim L^{\frac{d}{2}}$ independent samples $\{a^{(n)}\}_{n=1,\cdots,N}$ from $\langle\cdot\rangle_L$

$$\Big\langle\Big|\frac{1}{N}\sum_{n=1}^{N}\bar{a}_L^{(n)} - \bar{a}\Big|^2\Big\rangle_L \lesssim L^{-2d}\ln^d L,$$

$$\Big\langle\Big|\frac{L^d}{N-1}\sum_{m=1}^{N}\Big(\bar{a}_L^{(m)} - \frac{1}{N}\sum_{n=1}^{N}\bar{a}_L^{(n)}\Big)^{\otimes 2} - \bar{Q}\Big|^2\Big\rangle_L \lesssim L^{-d}\ln^d L$$

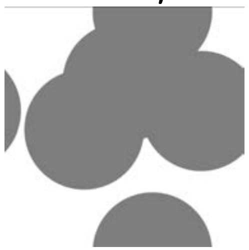
... at no further cost than $\bar{a}$

Back to numerical example

$N \sim L^{\frac{d}{2}}$ independent samples $\{a^{(n)}\}_{n=1,\cdots,N}$ from $\langle\cdot\rangle_L$,

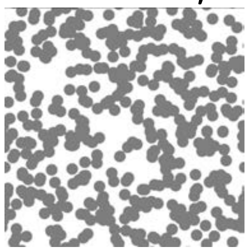
$$\left\langle \left| \frac{L^d}{N-1} \sum_{m=1}^{N} (\bar{a}_L^{(m)} - \frac{1}{N}\sum_{n=1}^{N} \bar{a}_L^{(n)})^{\otimes 2} - \bar{Q} \right|^2 \right\rangle_L \lesssim L^{-d} \ln^d L$$

L=2, N=500



$$\bar{Q} = 10^{-2} \times \begin{pmatrix} 1.01 & 0.00 & 0.00 & 0.09 \\ 0.00 & 0.31 & 0.09 & 0.00 \\ 0.00 & 0.09 & 0.31 & 0.00 \\ 0.09 & 0.00 & 0.00 & 0.99 \end{pmatrix}$$

L=20, N=500



$$\bar{Q} = 10^{-2} \times \begin{pmatrix} 1.00 & 0.00 & 0.00 & 0.23 \\ 0.00 & 0.56 & 0.23 & 0.00 \\ 0.00 & 0.23 & 0.56 & 0.00 \\ 0.23 & 0.00 & 0.00 & 1.01 \end{pmatrix}$$

Higher order comes naturally, i. e. 2nd order

2nd-order two-scale expansion: $u \approx (1+\phi_i\partial_i+\phi'_{ij}\partial_{ij})\bar{u}'$,
where $\bar{u}' := \bar{u} + \tilde{u}'$ with $\nabla\cdot(\bar{a}\nabla\tilde{u}' + \bar{a}'_i\nabla\partial_i\bar{u}) = 0$
and tensor $\bar{a}'_i$ is 2nd-order homogenized coefficient.

k-the component of 2nd-order commutator:
$\Xi'_k[u] := e_k\cdot(a-\bar{a})\nabla u + \bar{a}^{*\prime}_k e_l\cdot\nabla\partial_l u$,
characterized by $\Xi'_k[u] = \nabla^2 :$ something for a-harmonic u

Inject: $\Xi^{0\prime}[\bar{u}'](x) := \Xi'\big[(1+\phi_i\partial_i+\phi'_{ij}\partial_{ij})T'_x\bar{u}'\big](x)$,
where $T'_x\bar{u}'$ is 2nd-order *Taylor polynomial* of $\bar{u}'$ at x

A relative error of $O(L^{-\frac{d}{2}})$

Recipe: Inject two-scale expansion $(1+\phi_i\partial_i+\phi'_{ij}\partial_{ij})\bar u'$
into commutator $\Xi'_k[u] := e_k\cdot(a-\bar a)\nabla u + \bar a^{*\prime}_k e_l\cdot\nabla\partial_l u$,
in sense of $\Xi^{0\prime}[\bar u'](x) := \Xi'\big[(1+\phi_i\partial_i+\phi'_{ij}\partial_{ij})T'_x\bar u'\big](x)$

Duerinckx&O.'18 ($d=3$):

$$L^d\mathrm{var}\Big(\int g\cdot\nabla u - \int\nabla\bar v'\cdot\Xi'[u]\Big)$$
$$+L^d\mathrm{var}\Big(\int g\cdot\Xi'[u] - \int g\cdot\Xi^{0\prime}[\bar u']\Big) \le C(d,\lambda)L^{-d}$$

where $\bar v' = \bar v + \tilde v'$ with $\nabla\cdot(\bar a^*\nabla\tilde v' + a^{*\prime}_k\nabla\partial_k\bar v) = 0$

Relies on stochastic estimates of ϕ'_{ij} (Gu, Bella&Fehrman&Fischer&O)

Helpful tool: Flux correctors ...

1st-order: $ae_i = \bar{a}e_i - a\nabla\phi_i + \nabla\cdot\sigma_i$, σ_i skew,
2nd-order: $(\phi_i a - \sigma_i)e_j = \bar{a}_i' e_j - a\nabla\phi_{ij}' + \nabla\cdot\sigma_{ij}'$

2nd-order two-scale expansion; for $\bar{a}$-harmonic $\bar{u}$:
$\nabla\cdot a\nabla(1+\phi_i\partial_i+\phi_{ij}'\partial_{ij})\bar{u}' = \nabla\cdot\big((\phi_{ij}'a-\sigma_{ij}')\nabla\partial_{ij}\bar{u}'+\bar{a}_i'\nabla\partial_i\tilde{u}'\big)$,

2nd-order commutator; for a^*-harmonic u:
$e_k\cdot(a-\bar{a})\nabla u + {a_k^*}'e_l\cdot\nabla\partial_l u = \partial_l\nabla\cdot\big(({\phi_{kl}^*}'a+{\sigma_{kl}^*}')\nabla u\big)$

stochastic estimates: σ_i, σ_{ij}' like ϕ_i, ϕ_{ij} (Gloria&Neukamm&O.'13)

... bring residue in divergence-form

Helpful tool: Malliavin calculus and spectral gap

Spectral gap: $\mathrm{var}(F) \le \langle \int |\frac{\partial F}{\partial a}|^2 dx \rangle$

random variable F = functional of a,

Malliavin-derivative = functional derivative

For 1st-order $F := \int g \cdot (a - \bar{a})(\nabla u - \partial_i \bar{u}(e_i + \nabla \phi_i))$

Crucial formula: δa infinitesimal perturbation of a

$$\delta\Big(e_j \cdot (a - \bar{a})\big(\nabla u - \partial_i \bar{u}(e_i + \nabla \phi_i)\big)\Big)$$

$$= \underbrace{(e_j + \nabla \phi_j^*)}_{O(1)} \cdot \delta a \Big(\underbrace{\nabla u - \partial_i \bar{u}(e_i + \nabla \phi_i)}_{O(L^{-1}\nabla \bar{u})} \Big)$$

$$- \underbrace{\nabla}_{\rightsquigarrow O(L^{-1})} \cdot \Big(\underbrace{(\phi_j^* a + \sigma_j^*)}_{O(1)} \underbrace{\nabla \delta u}_{O(\nabla \bar{u}\, \delta a)} \Big) + \partial_i \bar{u} \nabla \cdot \Big((\phi_j^* a + \sigma_j^*) \underbrace{\nabla \delta \phi_i}_{O(\delta a)} \Big)$$

$$- \nabla \cdot (\phi_j^* \delta a \nabla u) + \partial_i \bar{u} \nabla \cdot (\phi_j^* \delta a (e_i + \nabla \phi_i))$$

Credits

Gaussianity of various errors: Nolen'14 based on Stein/Chatterjee, Biskup&Salvi&Wolf'14, Rossignol'14, ...

Quartic tensor Q via Helffer-Sjöstrand and Mahrarens& O.'13: Mourrat&O'14, Gu&Mourrat'15

Heuristics of a path-wise approach w/o Ξ: Gu&Mourrat'15, based on variational approach by Armstrong&Smart '13

$\nabla\varphi = \bar{a}$-Helmholtz-projection of white noise: Armstrong&Mourrat&Kuusi'16, Gloria&O.'16 based on finite range rather than Spectral Gap

Stochastic homogenization $\leftrightarrow$ SPDE

Parabolic PDE with singular driver
and rough coefficients

H. Weber, J. Sauer, S. Smith

The singular case

Nonlinear parabolic $\partial_t u - \mathrm{tr}(a(u)\mathrm{D}_x^2 u) = \sigma(u) f$
with rough "driver" f (eg. space-time white noise)

What means rough? parabolic Carnot-Caratheodory metric

$d((x,t),(y,s)) := |x-y| + \sqrt{|s-t|}$,

$[u]_\alpha := \sup_{(x,t)\neq(y,s)} \frac{|u(y,s)-u(x,t)|}{d^\alpha((y,s),(x,t))}$ for $0 < \alpha < 1$.

Hölder scale: $C^\alpha := \{[u]_\alpha < \infty\}$, $C^{2-\alpha} := \partial_1^2 C^\alpha + \partial_2 C^\alpha$.

$f \in C^{\alpha-2} \overset{\text{in best case}}{\Longrightarrow} \partial_1^2 u \in C^{\alpha-2},\ \sigma(u), a(u) \in C^\alpha$.

products $a(u)\partial_1^2 u$, $\sigma(u) f$ make sense
only if $\alpha + (\alpha-2) > 0 \iff \alpha > 1$.

Rough path approach

Semi-linear case $\partial_t u - \triangle u = \sigma(u) f$

Hairer-Pardoux '15 $\alpha \in (\frac{2}{5}, \frac{1}{2})$ (via "regularity structures")

Quasilinear case $\partial_t u - \mathrm{tr}(a(u) \mathrm{D}_x^2 u) = \sigma(u) f$

O.&Weber '16 (infinite-dimensional "model"),
Furlan&Gubinelli '16 (& para-calculus),
Bailleul&Debussche&Hofmanova '16 (para-calculus); all $\alpha \in (\frac{2}{3}, 1)$
Hairer&Gerencser '17 (regularity structures),
O.&Weber&Sauer&Smith '18; all $\alpha \in (\frac{1}{2}, \frac{2}{3})$

Our PDE approach:

Linear with *rough coefficient* $\partial_t u - \mathrm{tr}(a \mathrm{D}^2 u) = f$

Connections SPDE/homogenization

Linear parabolic $\partial_t u - \mathrm{tr}(a\mathrm{D}^2 u) = f$ with

singular rhs $f \in C^{\alpha-2}$, rough coefficient $a \in C^{\alpha}$, $\alpha \in (0,1)$

Given f, understand regularity of (nonlinear) map $a \mapsto u$

(on level of "modelled distributions")

Fight against roughness on small scales/

capitalize on cancellations on large scales

Augmented Schauder theory (polynomials + rough paths)

via kernel-free approach of Safonov/

Borrow (Schauder) regularity theory of $-\nabla \cdot \bar{a}\nabla$

via Campanato iteration (Avellaneda & Lin)

Algebraic: regularity structures / higher-order correctors