

Colloquium du CERMICS



Reduced order model approach for imaging with waves

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Motivation: Sensor array imaging

- Sensor array imaging (echography in medical imaging, sonar, non-destructive testing, seismic exploration, etc) has two steps:
 - data acquisition: an unknown medium is probed with waves; waves are emitted by a source (or a source array) and recorded by a receiver (or a receiver array).
 - data processing: the recorded signals are processed to identify the quantities of interest (reflector locations, etc).

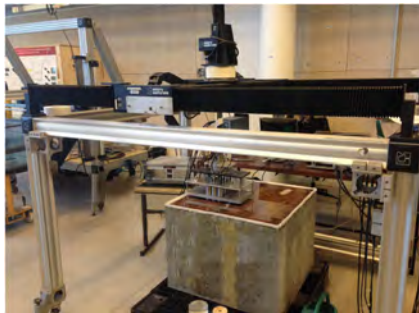
Example:

Ultrasound echography

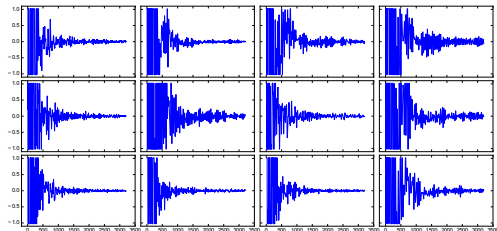


- Mathematically: Ill-posed inverse problems.

Example: Ultrasound in concrete

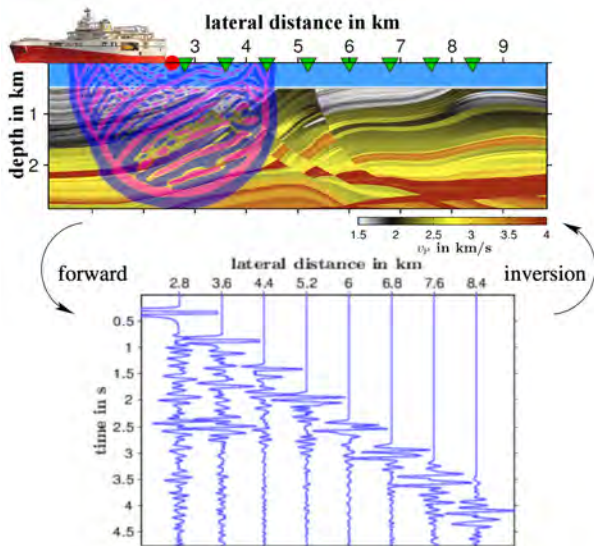


Experience: nondestructive testing



Data: recorded signals

Example: Reflection seismology



Velocity estimation problem

- *Direct problem:* Given the velocity map $c = (c(x))_{x \in \Omega}$ compute the wavefield solution of the wave equation

$$[\partial_t^2 - c^2(x)\Delta]p^{(s)}(t, x) = f(t)\delta(x - x_s), \quad t \in \mathbb{R}, \quad x \in \Omega \subset \mathbb{R}^d,$$

starting from $p^{(s)}(t, x) = 0$, $t \ll 0$, + boundary conditions at $\partial\Omega$.
At the locations of the receivers:

$$d_{r,s}(t) = p^{(s)}(t, x_r), \quad r, s = 1, \dots, N$$

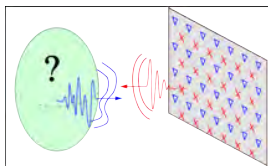
\hookrightarrow forward map

$$\mathcal{D} : c \mapsto \mathbf{d}$$

where $\mathbf{d} = ((d_{r,s}(t))_{r,s=1}^N)_{t \in [t_{\min}, t_{\max}]}$, is the array response matrix.

- *Inverse problem:*

Given the time-resolved measurements \mathbf{d} ,
determine the velocity map c .



Full Waveform Inversion (FWI)

- FWI fits data with the model prediction (least-square minimization):

$$\hat{c} = \underset{c}{\operatorname{argmin}} \mathcal{O}_{FWI}[c],$$

$$\mathcal{O}_{FWI}[c] = \|\mathcal{D}[c] - \mathbf{d}_{meas}\|^2 = \sum_{r,s=1}^N \int_{t_{\min}}^{t_{\max}} |\mathcal{D}[c](t)_{r,s} - d_{meas}(t)_{r,s}|^2 dt$$

Cf [Virieux and Operto 2009].

- Problem: The objective function $\mathcal{O}_{FWI}[c]$ is not convex in c .
↪ optimization needs hard to get good initial guess.

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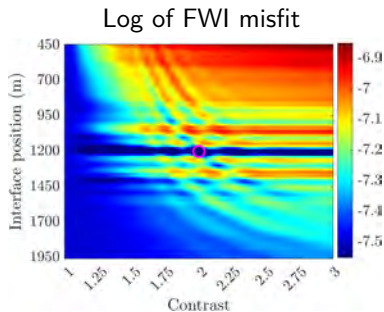
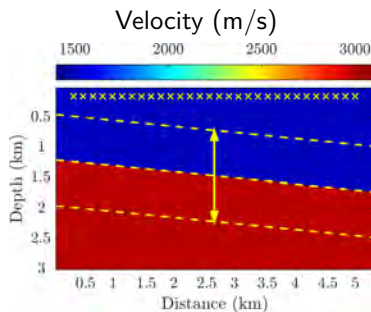
$$\hat{c} = \operatorname{argmin}_c \mathcal{O}_{FWI}[c],$$

$$\mathcal{O}_{FWI}[c] = \|\mathcal{D}[c] - \mathbf{d}_{meas}\|^2 = \sum_{r,s=1}^N \int_{t_{\min}}^{t_{\max}} |\mathcal{D}[c](t)_{r,s} - d_{meas}(t)_{r,s}|^2 dt$$

Cf [Virieux and Operto 2009].

- Problem: The objective function $\mathcal{O}_{FWI}[c]$ is not convex in c .
 \hookrightarrow optimization needs hard to get good initial guess.
- Regularization: $\hat{c} = \operatorname{argmin}_c \{\mathcal{O}_{FWI}[c] + \lambda \operatorname{Reg}[c]\}$, with $\operatorname{Reg}[c] = \|c\|_{L^2}^2, \|c\|_{L^1}, \|c\|_{TV}, \dots$ (Bayesian interpretation).
- Layer stripping: Proceed hierarchically from the shallow part to the deep part [Wang et al. 2009]
- Frequency hopping: Successive inversion of subdata sets of increasing high-frequency content [Bunks et al. 1995]
- Optimal transport: Wasserstein distance instead of least-squares [Engquist et al., 2014, Métivier et al. 2016]

Topography of the FWI objective function



- Search velocity has two parameters: the bottom velocity and depth of the interface (the angle and top velocity are known).
- Probing pulse is a modulated Gaussian pulse with central frequency 6 Hz and bandwidth 4 Hz ($\lambda \simeq 300\text{ m}$ at 10 Hz).
 $N = 30$ sensors; $N_t = 39$ time samples at interval $\tau = 0.0435\text{ s}$.
- Objective function:

$$\mathcal{O}_{FWI}[c] = \|\mathcal{D}[c] - \mathbf{d}_{meas}\|^2$$

↪ Many local minima.

Objective

- Objective: Find a convex formulation of FWI.
- Proposed approach: find a (nonlinear) mapping $\mathcal{R}(\mathbf{d})$ such that:

$$\mathcal{O}[c] = \|\mathcal{R}(\mathcal{D}[c]) - \mathcal{R}(\mathbf{d}_{meas})\|^2$$

has better convexity properties than

$$\mathcal{O}_{FWI}[c] = \|\mathcal{D}[c] - \mathbf{d}_{meas}\|^2$$

Remark : We can think of the mapping \mathcal{R} as a nonlinear preconditioner of the forward mapping \mathcal{D} .

Towards the ROM objective function

- *Ideal objective function 1:*

$$\mathcal{O}[c] = \|c - c_{meas}\|^2 = \int_{\Omega} |c(x) - c_{meas}(x)|^2 dx$$

but c_{meas} is not observed (i.e., cannot be extracted from \mathbf{d}_{meas}) !

Towards the ROM objective function

- Let us consider the wave operator

$$\mathcal{A}[c] = -c(x)\Delta[c(x) \cdot]$$

- Galerkin method to approximate the operator \mathcal{A} by a matrix:
 - consider a space of (piecewise polynomial) functions with basis $(\Psi_l(x))_{l=1}^L$,
 - consider the row vector field $\Psi(x) = (\Psi_1(x), \dots, \Psi_L(x))$ and define:

$$\mathbf{A}^\Psi = \int_{\Omega} dx \Psi(x)^T \mathcal{A} \Psi(x) \in \mathbb{R}^{L \times L}$$

- *Ideal objective function 2:*

$$\mathcal{O}[c] = \|\mathbf{A}^\Psi[c] - \mathbf{A}_{meas}^\Psi\|^2 = \sum_{l,l'=1}^L |A^\Psi[c]_{ll'} - A_{meas, ll'}^\Psi|^2$$

but \mathbf{A}_{meas}^Ψ is not observed !

The ROM matrix

- Our Galerkin approximation space:

- consider a time discretization $\{t_j = j\tau\}_{0 \leq j < N_t}$ with uniform stepping τ ,
- gather the waves $p^{(s)}(t, x)$ evaluated at $t = t_j$ for all the N sources:

$$\mathbf{p}_j(x) = \left(p^{(1)}(t_j, x), \dots, p^{(N)}(t_j, x) \right), \quad x \in \Omega.$$

(note: apply first a linear preprocessing).

- organize the first N_t snapshots in the NN_t dimensional row vector field:

$$\mathbf{U}(x) = (\mathbf{p}_0(x), \dots, \mathbf{p}_{N_t-1}(x)), \quad x \in \Omega.$$

- apply Gram-Schmidt orthogonalization onto $\mathbf{U}(x) = \mathbf{V}(x)\mathbf{R}$.

- Define ROM matrix:

$$\mathbf{A}^{rom} = \int_{\Omega} dx \mathbf{V}(x)^T \mathcal{A} \mathbf{V}(x) \in \mathbb{R}^{NN_t \times NN_t}.$$

- *Ideal objective function 3:*

$$\mathcal{O}[c] = \|\mathbf{A}^{rom}[c] - \mathbf{A}_{meas}^{rom}\|^2$$

but \mathbf{A}_{meas}^{rom} is not observed (neither \mathcal{A} nor $\mathbf{V}(x)$ is observed) !

The ROM matrix

- Our Galerkin approximation space:
 - consider a time discretization $\{t_j = j\tau\}_{0 \leq j < N_t}$ with uniform stepping τ ,
 - gather the waves $p^{(s)}(t, x)$ evaluated at $t = t_j$ for all the N sources:

$$\mathbf{p}_j(x) = \left(p^{(1)}(t_j, x), \dots, p^{(N)}(t_j, x) \right), \quad x \in \Omega.$$

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- apply Gram-Schmidt orthogonalization onto $\mathbf{U}(x) = \mathbf{V}(x)\mathbf{R}$.

- Define ROM matrix:

$$\mathbf{A}^{rom} = \int_{\Omega} dx \mathbf{V}(x)^T \mathcal{A} \mathbf{V}(x) \in \mathbb{R}^{NN_t \times NN_t}.$$

- **Proposition:** *The ROM matrix \mathbf{A}^{rom} can be extracted from the measurements \mathbf{d} , without knowing \mathcal{A} nor $\mathbf{V}(x)$.*

$\hookrightarrow \mathcal{O}_{ROM}[c] = \|\mathbf{A}^{rom}[c] - \mathbf{A}_{meas}^{rom}\|^2$ is a legitimate objective function.

Proof of the proposition

First step: *Linear preprocessing.*

- Define the new data matrix $\mathbf{d}^f(t)$:

$$\mathbf{d}^f(t) = [-f'(-\cdot) *_t \mathbf{d}](t) + [-f'(-\cdot) *_t \mathbf{d}](-t).$$

Second step: *Expression of the new data entries as wave correlations.*

- Introduce the solution $u^{(s)}(t, x)$ of the homogeneous wave equation

$$(\partial_t^2 + \mathcal{A})u^{(s)}(t, x) = 0, \quad t > 0, \quad x \in \Omega,$$

with boundary conditions on $\partial\Omega$, with initial state

$$u^{(s)}(0, x) = u_0^{(s)}(x) = \left| \hat{f}(\sqrt{\mathcal{A}}) \right| \delta(x - x_s), \quad \partial_t u^{(s)}(0, x) = 0.$$

It has the form

$$u^{(s)}(t, x) = \cos(t\sqrt{\mathcal{A}})u_0^{(s)}(x).$$

→ The entries of $\mathbf{d}^f(t)$ can be expressed as wave correlations:

$$d_{r,s}^f(t) = \int_{\Omega} dx u_0^{(r)}(x)u^{(s)}(t, x).$$

Proof of the proposition

Third step: *Definition of the ROM.*

Let $\tau > 0$ be fixed.

- Gather the N_t snapshots for all the N sources in the row vector fields

$$\mathbf{u}_j(x) = \left(u^{(1)}(j\tau, x), \dots, u^{(N)}(j\tau, x) \right), \quad 0 \leq j < N_t.$$

- Organize the first N_t snapshots in the NN_t dimensional row vector field:

$$\mathbf{U}(x) = (\mathbf{u}_0(x), \dots, \mathbf{u}_{N_t-1}(x)), \quad x \in \Omega.$$

- Apply Gram-Schmidt orthogonalization onto $\mathbf{U}(x) = \mathbf{V}(x)\mathbf{R}$.

(note: we have $\int_{\Omega} dx \mathbf{V}(x)^T \mathbf{V}(x) = \mathbf{I}_{NN_t}$).

- Define

$$\mathbf{A}^{rom} = \int_{\Omega} dx \mathbf{V}(x)^T \mathcal{A} \mathbf{V}(x)$$

Proof of the proposition

Fourth step: Expression of the ROM in terms of mass and stiffness.

- Define the $NN_t \times NN_t$ “mass” and “stiffness” matrices:

$$\mathbf{M} = \int_{\Omega} dx U^T(x)U(x), \quad \mathbf{S} = \int_{\Omega} dx U^T(x)\mathcal{A}U(x)$$

- Since $U(x) = V(x)\mathbf{R}$, we get

$$\begin{aligned}\mathbf{M} &= \mathbf{R}^T \int_{\Omega} dx V^T(x)V(x)\mathbf{R} \\ &= \mathbf{R}^T\mathbf{R}\end{aligned}$$

and

$$\begin{aligned}\mathbf{A}^{rom} &= \int_{\Omega} dx V(x)^T \mathcal{A}V(x) = \mathbf{R}^{-T} \int_{\Omega} dx U(x)^T \mathcal{A}U(x)\mathbf{R} \\ &= \mathbf{R}^{-T}\mathbf{S}\mathbf{R}\end{aligned}$$

$\Leftrightarrow \mathbf{A}^{rom}$ can be expressed in terms of \mathbf{M} and \mathbf{S} .

Proof of the proposition

Fifth step: *Expression of the ROM in terms of data.*

The $N \times N$ blocks of the mass matrix \mathbf{M} are

$$\begin{aligned}\mathbf{M}_{i,j} &= \langle \mathbf{u}_i, \mathbf{u}_j \rangle_{L^2(\Omega)} = \langle \cos(i\tau\sqrt{\mathcal{A}})\mathbf{u}_0, \cos(j\tau\sqrt{\mathcal{A}})\mathbf{u}_0 \rangle_{L^2(\Omega)} \\ &= \langle \mathbf{u}_0, \cos(i\tau\sqrt{\mathcal{A}})\cos(j\tau\sqrt{\mathcal{A}})\mathbf{u}_0 \rangle_{L^2(\Omega)} \\ &= \frac{1}{2} \langle \mathbf{u}_0, [\cos((i+j)\tau\sqrt{\mathcal{A}}) + \cos(|i-j|\tau\sqrt{\mathcal{A}})]\mathbf{u}_0 \rangle_{L^2(\Omega)} \\ &= \frac{1}{2} \langle \mathbf{u}_0, \mathbf{u}_{i+j} + \mathbf{u}_{|i-j|} \rangle_{L^2(\Omega)} \\ &= \frac{1}{2} \left(\mathbf{d}^f((i+j)\tau) + \mathbf{d}^f(|i-j|\tau) \right), \quad 0 \leq i, j < N_t.\end{aligned}$$

Idem for the stiffness matrix \mathbf{S} .

↪ \mathbf{M} and \mathbf{S} can be expressed in terms of the data matrix \mathbf{d}^f .

Algorithm for data-driven ROM matrix

Input: The matrices $\mathbf{d}(t) = (d_{r,s}(t))_{r,s=1}^N$ of measurements.

1. Compute $d_{r,s}^f(t) = [-f'(-\cdot) *_t d_{r,s}](t) + [-f'(-\cdot) *_t d_{r,s}](-t)$ and

$$\mathbf{D}_j = \mathbf{d}^f(j\tau), \quad 0 \leq j \leq 2N_t - 2.$$

2. Compute $\mathbf{D}_j = \mathbf{d}^f(j\tau)$, for $j = 0, \dots, 2N_t - 2$ with $\ddot{d}_{r,s}^f(t) = \partial_t^2 d_{r,s}^f(t)$ using, e.g., the Fourier transform.

3. Calculate $\mathbf{M}, \mathbf{S} \in \mathbb{R}^{NN_t \times NN_t}$ with the block entries

$$\mathbf{M}_{i,j} = \frac{1}{2}(\mathbf{D}_{i+j} + \mathbf{D}_{|i-j|}) \in \mathbb{R}^{N \times N},$$

$$\mathbf{S}_{i,j} = -\frac{1}{2}(\mathbf{D}_{i+j} + \mathbf{D}_{|i-j|}) \in \mathbb{R}^{N \times N},$$

for $0 \leq i, j \leq N_t - 1$.

4. Perform block Cholesky factorization $\mathbf{M} = \mathbf{R}^T \mathbf{R}$.

Output: $\mathbf{A}^{rom} = \mathbf{R}^{-T} \mathbf{S} \mathbf{R}^{-1}$.

ROM objective function

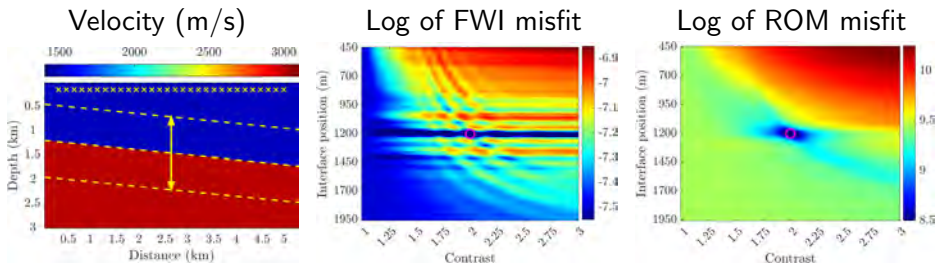
- ROM misfit function:

$$\mathcal{O}_{ROM}[c] = \|\mathbf{A}^{rom}[c] - \mathbf{A}_{meas}^{rom}\|^2$$

where $\mathbf{A}^{rom}[c]$ is computed from $\mathcal{D}[c]$ and \mathbf{A}_{meas}^{rom} is computed from \mathbf{d}_{meas} .

- For a rich enough space of snapshots, the ROM matrix \mathbf{A}^{rom} contains roughly the same information as $\mathcal{A} = -c(x)\Delta[c(x) \cdot]$.
↔ The ROM misfit function should have nice convexity properties.
- Conjecture: “rich enough” would mean for sensors all around the domain of interest, separated by roughly half a wavelength, for time sampling satisfying the Nyquist criterium.
↔ Conjecture proved only in special situations.

Topographies of the FWI and ROM objective functions



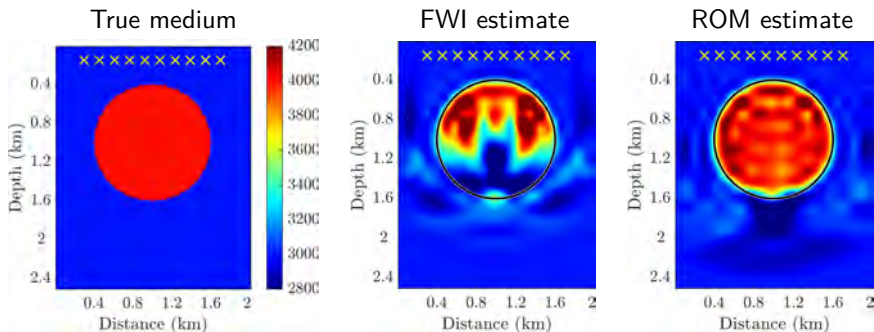
- Search velocity has two parameters: the contrast and the depth of the interface (the angle and top velocity are known).
- FWI objective function:

$$\mathcal{O}_{FWI}[c] = \|\mathcal{D}[c] - \mathbf{d}_{meas}\|^2$$

- ROM objective function:

$$\mathcal{O}_{ROM}[c] = \|\mathbf{A}^{rom}[c] - \mathbf{A}_{meas}^{rom}\|^2$$

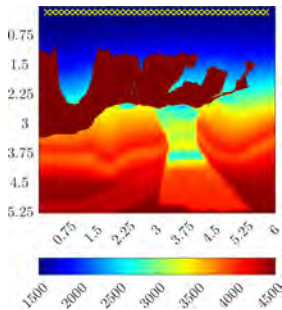
Camembert model



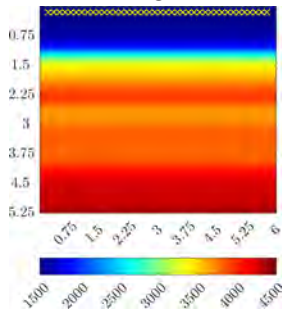
- Probing pulse is a modulated Gaussian pulse with central frequency 6Hz and bandwidth 4Hz ($\lambda = 300\text{m}$ at 10Hz).
- Search velocity: $c(x, \boldsymbol{\eta}) = c_o + \sum_l \eta_l \phi_l(x)$, $\boldsymbol{\eta} = (\eta_l)_{l=1}^L$.
- $\phi_l(x)$ are Gaussian peaks with centers on a regular grid, $L = 400$, with width 60m (0.2λ).
- FWI minimizes $\mathcal{O}_{FWI}(\boldsymbol{\eta}) = \|\mathcal{D}[c(\boldsymbol{\eta})] - \mathbf{d}_{meas}\|^2 + \mu\|\boldsymbol{\eta}\|^2$
- ROM minimizes $\mathcal{O}_{ROM}(\boldsymbol{\eta}) = \|\mathbf{A}^{rom}[c(\boldsymbol{\eta})] - \mathbf{A}_{meas}^{rom}\|^2 + \mu\|\boldsymbol{\eta}\|^2$

Salt body (BP - model)

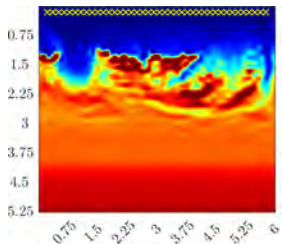
True model



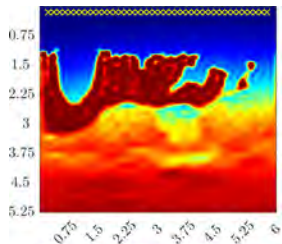
Initial guess



FWI estimate



ROM estimate



One limitation and two extensions

- One limitation of the method:

We need co-located sources and receivers.

- Extension to passive imaging:

Consider a *receiver* array at $(x_r)_{r=1}^N$ recording signals transmitted by noise sources (uncontrolled, opportunistic sources).

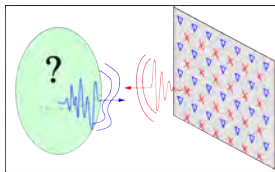
Compute the cross correlation matrix of the recorded signals.

→ The ROM procedure is natural in the passive framework, since the cross correlation matrix gives directly the data matrix

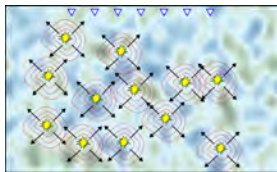
$$\mathbf{d}^f(t) = (d_{r,r'}^f(t))_{r,r'=1}^N.$$

- Extension to vector waves.

Passive imaging



Active acquisition



Passive acquisition

- Consider the solution $p(t, x)$ of the wave equation

$$\partial_t^2 p - c^2(x)\Delta p = s(t, x), \quad t \in \mathbb{R}, \quad x \in \Omega \subset \mathbb{R}^d,$$

where $s(t, x)$ is a zero-mean, stationary in time random process with

$$\langle s(t_1, y_1)s(t_2, y_2) \rangle = F(t_1 - t_2)K(y_1)\delta(y_1 - y_2)$$

- The passive data set is $((p(t, x_r))_{r=1}^N)_{t \in [0, T]}$, with $T \gg 1$.
- The empirical cross correlation of the recorded waves at x_r and $x_{r'}$ is

$$C_T(\tau, x_r, x_{r'}) = \frac{1}{T} \int_0^T dt p(t, x_r)p(t + \tau, x_{r'})$$

Passive imaging

- The statistical cross correlation

$$C^{(1)}(\tau, x_r, x_{r'}) = \langle C_T(\tau, x_r, x_{r'}) \rangle$$

is independent of T by stationarity of the noise sources.

- The statistical stability follows from the ergodicity of the noise sources:

$$C_T(\tau, x_r, x_{r'}) \xrightarrow{T \rightarrow +\infty} C^{(1)}(\tau, x_r, x_{r'}),$$

in probability [Garnier et al. 2016].

- **Proposition.** We have, for any $r, r' = 1, \dots, N$,

$$\partial_\tau^2 C^{(1)}(\tau, x_r, x_{r'}) = -\frac{1}{4} d_{r,r'}^f(\tau),$$

where $\mathbf{d}^f(t)$ is the active data matrix obtained with a source signal $f(t)$ that satisfies $|\hat{f}(\omega)| = \hat{F}(\omega)^{1/2}$ (\hat{F} = power spectral density of the noise sources).

- **Corollary.** *The passive data (cross correlation matrix) can be used in the ROM algorithm (no preprocessing).*

An extension to first-order systems

- General framework: The vectorial wave field $\psi_\varepsilon \in \mathbb{R}^m$ satisfies

$$\partial_t \psi_\varepsilon(t, \mathbf{x}) + \mathcal{L} \psi_\varepsilon(t, \mathbf{x}) = s(t) \mathbf{f}_\varepsilon(\mathbf{x}), \quad t \in \mathbb{R}, \mathbf{x} \in \Omega \subset \mathbb{R}^d,$$

where \mathcal{L} is skew-adjoint, $\mathbf{f}_\varepsilon(\mathbf{x})$ models a source localized at point $\mathbf{x}_{\varepsilon_1}$, with polarization indexed by ε_2 ($\varepsilon = (\varepsilon_1, \varepsilon_2) \in \mathbb{N}^2$).

- Can model acoustics, elasticity, and electromagnetism.
- The array response matrix is:

$$\mathbf{d}_{\varepsilon', \varepsilon}(t) = \int_{\Omega} d\mathbf{x} [\mathbf{f}_{\varepsilon'}(\mathbf{x})]^T \psi_\varepsilon(t, \mathbf{x}).$$

- Main hypothesis: Requires multi-dimensional, collocated sources and receivers.
- Main goal: Multiparametric inversion.

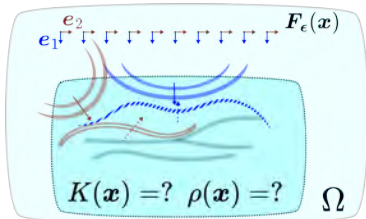
First-order acoustic system

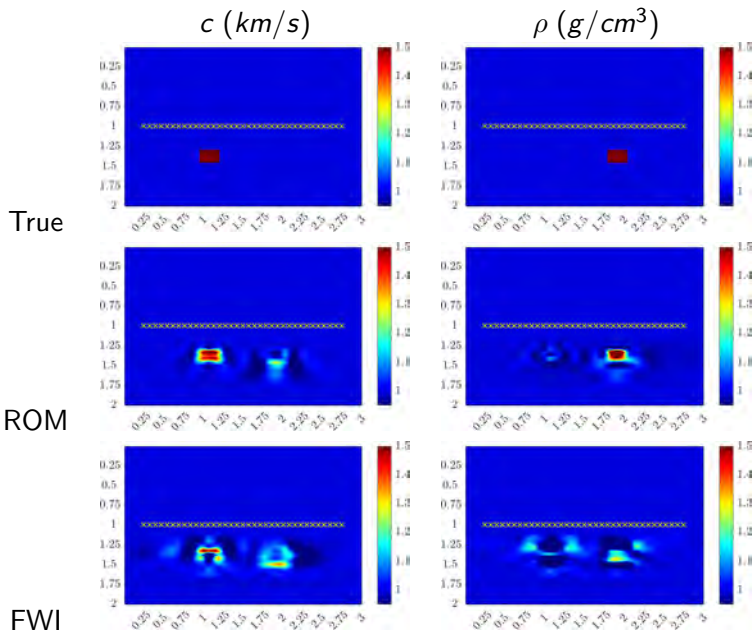
The acoustic wave equation:

$$\begin{aligned}\partial_t \mathbf{u}_\varepsilon(t, \mathbf{x}) + \rho^{-1}(\mathbf{x}) \nabla p_\varepsilon(t, \mathbf{x}) &= s(t) \mathbf{F}_\varepsilon(\mathbf{x}), \\ \partial_t p_\varepsilon(t, \mathbf{x}) + K(\mathbf{x}) \nabla \cdot \mathbf{u}_\varepsilon(t, \mathbf{x}) &= 0,\end{aligned}$$

can be formulated in the general first-order form with ($c = \sqrt{K/\rho}$):


$$\begin{aligned}\psi_\varepsilon(t, \mathbf{x}) &= \begin{pmatrix} \sqrt{\rho(\mathbf{x})} \mathbf{u}_\varepsilon(t, \mathbf{x}) \\ \frac{1}{\sqrt{K(\mathbf{x})}} p_\varepsilon(t, \mathbf{x}) \end{pmatrix}, & \mathbf{f}_\varepsilon(\mathbf{x}) &= \begin{pmatrix} \mathbf{F}_\varepsilon(\mathbf{x}) \\ 0 \end{pmatrix}, \\ \mathcal{L} &= \begin{pmatrix} 0 & \frac{1}{\sqrt{\rho(\mathbf{x})}} \text{grad} \left[c(\mathbf{x}) \sqrt{\rho(\mathbf{x})} \cdot \right] \\ c(\mathbf{x}) \sqrt{\rho(\mathbf{x})} \text{div} \left[\frac{1}{\sqrt{\rho(\mathbf{x})}} \cdot \right] & 0 \end{pmatrix}.\end{aligned}$$







Conclusions

- The ROM is an approximation of the wave operator on a space defined by the snapshots of the wavefield.
- This space is not known and neither is the wave operator.
- Yet, we can compute the ROM from the data !
- We can then formulate a velocity estimation algorithm that minimizes the ROM misfit and that avoids cycle skipping and other problems.
- The method can be applied to active and passive imaging.
- To be continued (for vector waves).

 L Borcea, J Garnier, AV Mamonov, J Zimmerling, When data driven reduced order modeling meets waveform inversion, *SIAM Rev.* 66 (3), 2024, 501-532.

 L Borcea, J Garnier, AV Mamonov, J Zimmerling, Waveform inversion with a data driven estimate of the internal wave, *SIAM Journal on Imaging Sciences* 16 (1), 2023, 280-312.

 L Borcea, J Garnier, AV Mamonov, J Zimmerling, Waveform inversion via reduced order modeling, *Geophysics* 88 (2), 2023, R175-R191.