Colloquium du CERMICS

Comment utiliser les non-linéarités pour contrôler un système

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Outline

1. Control systems: Examples and the controllability problem
2. Some controllability results in finite dimension
3. Return method, application to the control of the Euler equations
4. Scaling, application to the Navier-Stokes equations
5. Quasi-static deformations, application to a water tank control system

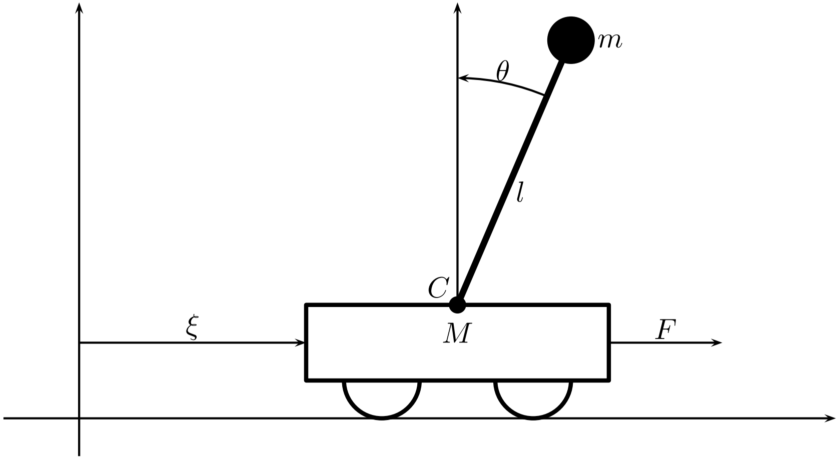
Notion of control systems

A control system in this talk is a **dynamical system** on which one can act by using suitable **controls**.
Mathematically it often takes the form

$$\Sigma : \quad \dot{y} = f(y, u),$$

where y is called the **state** and u is the **control**. The state can be in finite dimension (then $\dot{y} = f(y, u)$ is an ordinary differential equation) or in infinite dimension (example: $\dot{y} = f(y, u)$ is a partial differential equation). A solution $t \in [0, T] \mapsto (y(t), u(t))$ of $\dot{y}(t) = f(y(t), u(t))$ is called a **trajectory** of the control system Σ. A map $t \in [0, T] \mapsto y(t)$ is also called a trajectory of the control system Σ if there exists $t \in [0, T] \mapsto u(t)$ such that $\dot{y}(t) = f(y(t), u(t))$.

A first example: the cart-inverted pendulum



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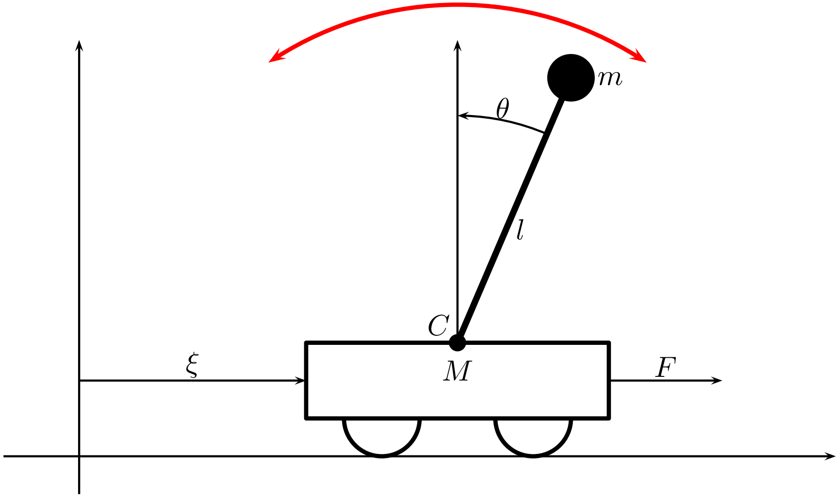


The cart-inverted pendulum: The equations

Let

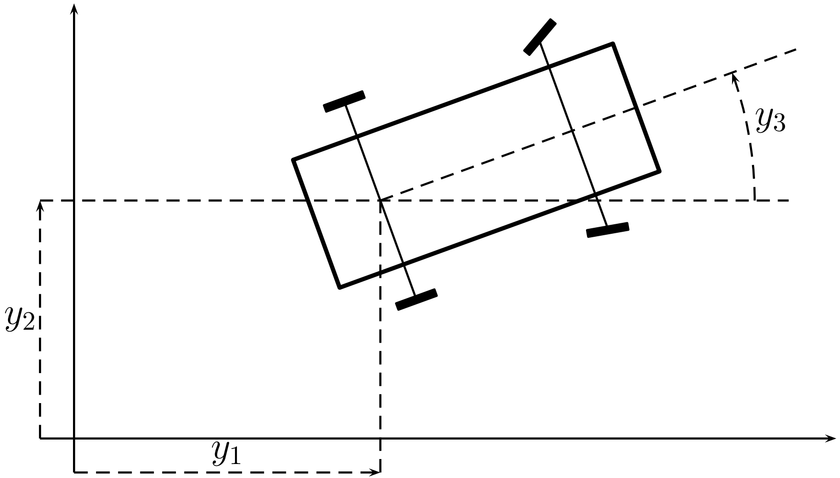
(1) $$y_1 := \xi,\ y_2 := \theta,\ y_3 := \dot{\xi},\ y_4 := \dot{\theta},\ u := F,$$

The dynamics of the cart-inverted pendulum system is $\dot{y} = f(y,u)$, with $y = (y_1, y_2, y_3, y_4)$ and

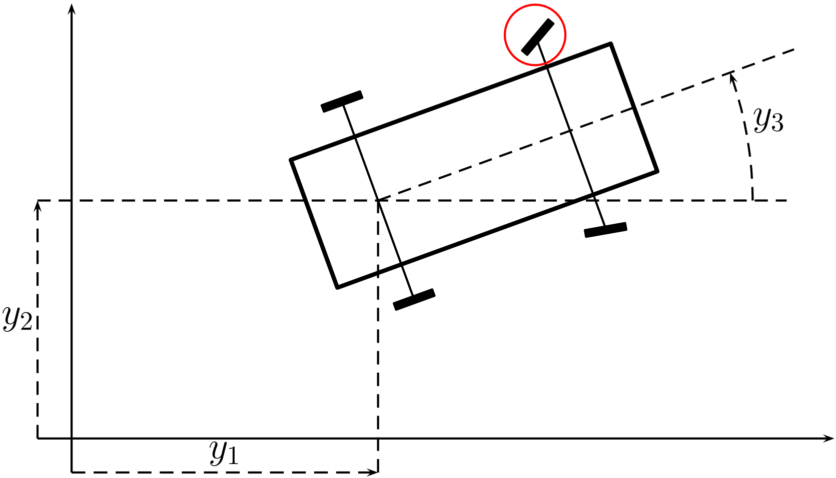
(2) $$f := \begin{pmatrix} y_3 \\ y_4 \\ \dfrac{mly_4^2 \sin y_2 - mg \sin y_2 \cos y_2}{M + m \sin^2 y_2} + \dfrac{u}{M + m \sin^2 y_2} \\ \dfrac{-mly_4^2 \sin y_2 \cos y_2 + (M+m) g \sin y_2 - u \cos y_2}{(M + m \sin^2 y_2) l} \end{pmatrix}.$$

The baby stroller

The baby stroller: The dynamic equations of motion



The baby stroller: The dynamic equations of motion



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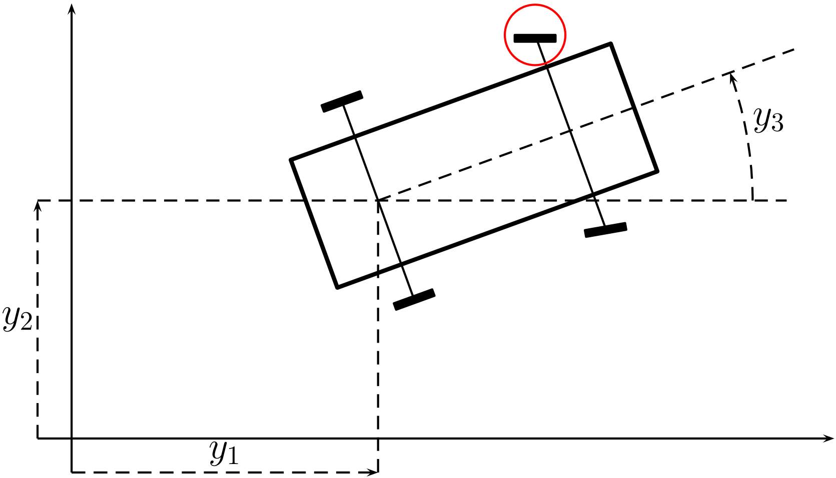
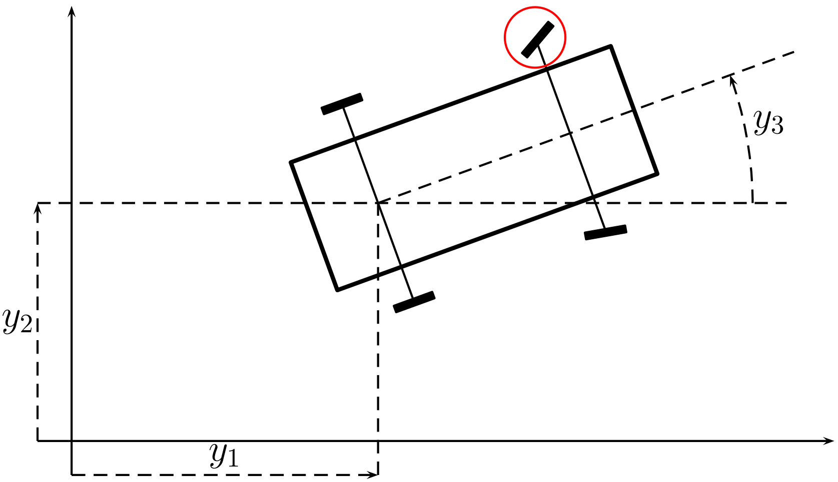
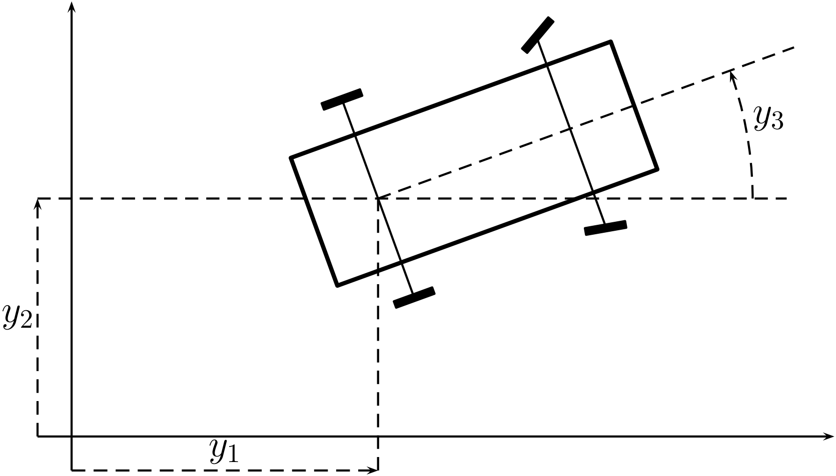


The baby stroller: The dynamic equations of motion



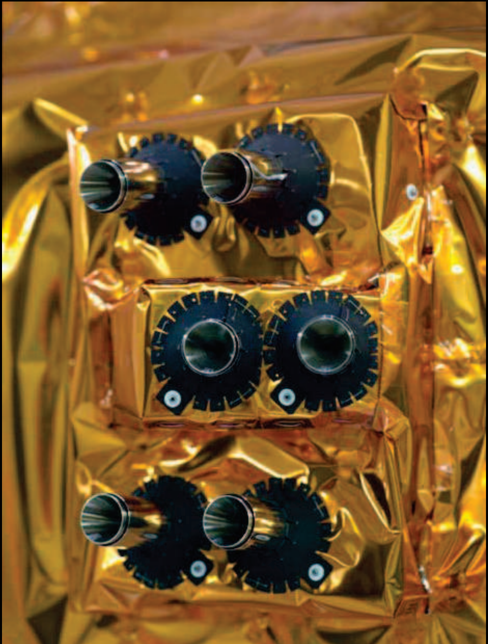
The baby stroller: The dynamic equations of motion



(1) $$\dot{y}_1 = u_1 \cos y_3,\ \dot{y}_2 = u_1 \sin y_3, \dot{y}_3 = u_2,\ n = 3,\ m = 2.$$

Satellite

Satellite : The thrusters



Control of the attitude: Notations

- $\eta = (\phi, \theta, \psi) \in \mathbb{R}^3$ are the Euler angles of a frame attached to the satellite representing rotations with respect to a fixed reference frame,
- $\omega = (\omega_1, \omega_2, \omega_3) \in \mathbb{R}^3$ is the angular velocity of the frame attached to the satellite with respect to the reference frame, expressed in the frame attached to the satellite,
- J is the inertia matrix of the satellite,
- The $b_1, \ldots, b_m$ are m fixed independent vectors in $\mathbb{R}^3$ and $u_i b_i \in \mathbb{R}^3$, $1 \leqslant i \leqslant m$, are the torques applied to the satellite, the $u_i \in \mathbb{R}$, $1 \leqslant i \leqslant m$, are the controls.

Control of the attitude: The dynamic equations of motion

(1) $$\dot{\omega} = J^{-1}S(\omega)J\omega + \sum_{i=1}^{m} u_i J^{-1} b_i, \quad \dot{\eta} = A(\eta)\omega,$$

where $S(\omega)$ is the matrix representation of the wedge-product, i.e.,

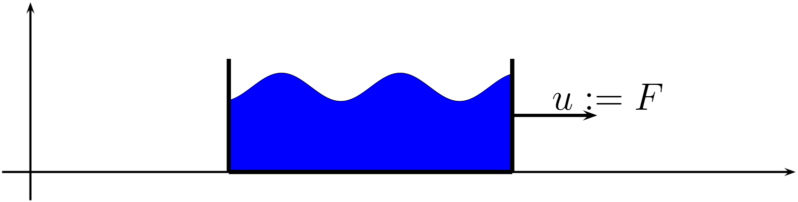
(2) $$S(\omega) = \begin{pmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{pmatrix},$$

and

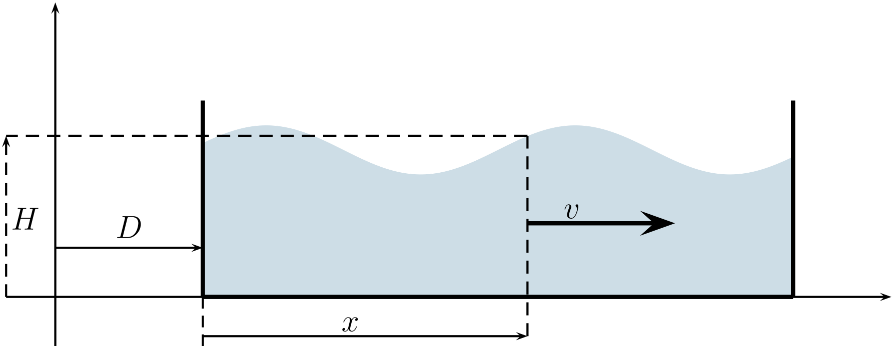
(3) $$A(\eta) = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ \sin\theta\tan\phi & 1 & -\cos\theta\tan\phi \\ -\sin\theta/\cos\phi & 0 & \cos\theta/\cos\phi \end{pmatrix}.$$

(Note that $A(0) = \mathrm{Id}$.) The state of our control system is $(\eta_1, \eta_2, \eta_3, \omega_1, \omega_2, \omega_3) \in \mathbb{R}^6$ and the control is $(u_1, \ldots, u_m) \in \mathbb{R}^m$.

A water-tank control system



Saint-Venant equations: Notations



The horizontal velocity v is taken with respect to the one of the tank.

The model: Saint-Venant equations

$$H_t + (Hv)_x = 0,\ t \in [0,T],\ x \in [0,L], \tag{1}$$

$$v_t + \left(gH + \frac{v^2}{2}\right)_x = -u\left(t\right),\ t \in [0,T],\ x \in [0,L], \tag{2}$$

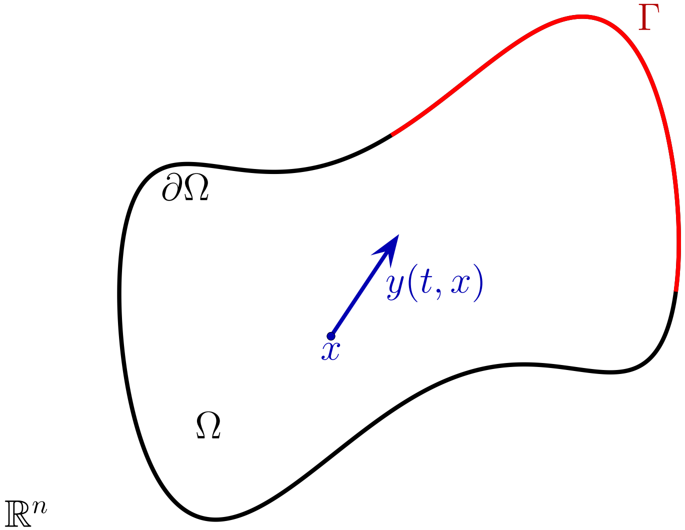
$$v(t,0) = v(t,L) = 0,\ t \in [0,T], \tag{3}$$

$$\dot{s}(t) = u\left(t\right),\ t \in [0,T], \tag{4}$$

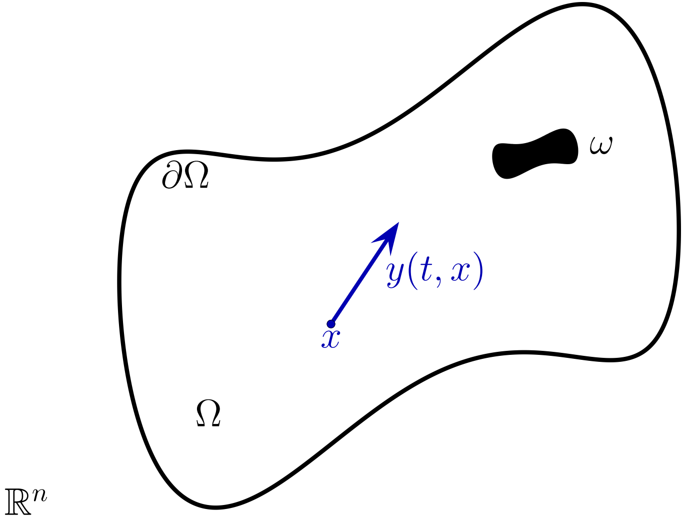
$$\dot{D}(t) = s\left(t\right),\ t \in [0,T]. \tag{5}$$

- $u\left(t\right)$ is the horizontal acceleration of the tank in the absolute referential,
- g is the gravity constant,
- s is the horizontal velocity of the tank,
- D is the horizontal displacement of the tank.

The Euler/Navier-Stokes control system



The Euler/Navier-Stokes control system



Euler control system

We denote by $\nu : \partial\Omega \to \mathbb{R}^n$ the outward unit normal vector field to Ω. Let $T > 0$. The Euler control system is

$$y_t + (y \cdot \nabla)y + \nabla p = 0, \text{ div } y = 0, \tag{1}$$

$$y \cdot \nu = 0 \text{ on } [0,T] \times (\partial\Omega \setminus \Gamma), \tag{2}$$

This is an implicit formulation. If one wants to make it explicit, many choices are in fact possible. For example, for $n = 2$, one can take

1. $y \cdot \nu$ on Γ with $\int_\Gamma y \cdot \nu = 0$,
2. $\text{curl } y$ at the points of $[0,T] \times \Gamma$ where $y \cdot \nu < 0$.

The Navier-Stokes control system

The Navier-Stokes control system is deduced from the Euler equations by adding the linear term $-\Delta y$: the equation is now

$$y_t - \Delta y + (y \cdot \nabla)y + \nabla p = 0, \text{ div } y = 0. \tag{1}$$

For the boundary condition, one requires now that

$$y = 0 \text{ on } [0, T] \times (\partial\Omega \setminus \Gamma). \tag{2}$$

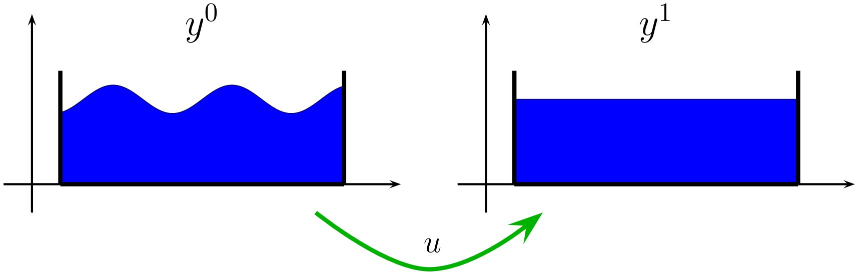
For the control, one can take, for example, y on $[0, T] \times \Gamma$.

Controllability

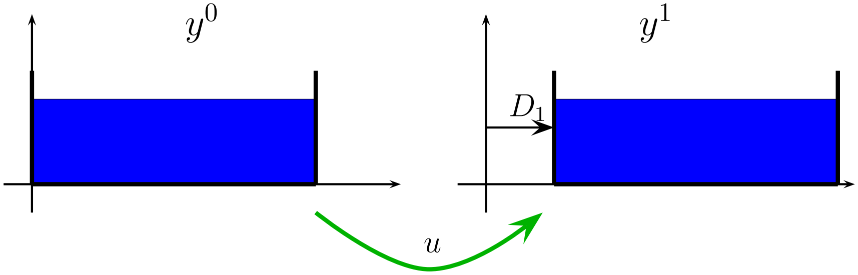
Given two states y^0 and y^1, does there exist a control $t \in [0,T] \mapsto u(t)$ which steers the control system from y^0 to y^1, i.e. such that

(1) $$\big(\dot{y} = f(y,u(t)),\ y(0) = y^0\big) \Rightarrow \big(y(T) = y^1\big)?$$

Example: Destroy waves



Another example: Steady-state controllability



Controllability of the Euler control system

Let y^0, $y^1 : \overline{\Omega} \to \mathbb{R}^n$ be such that

(1) $$\text{div } y^0 = \text{div } y^1 = 0,\ y^0 \cdot \nu = y^1 \cdot \nu = 0 \text{ on } \partial\Omega \setminus \Gamma.$$

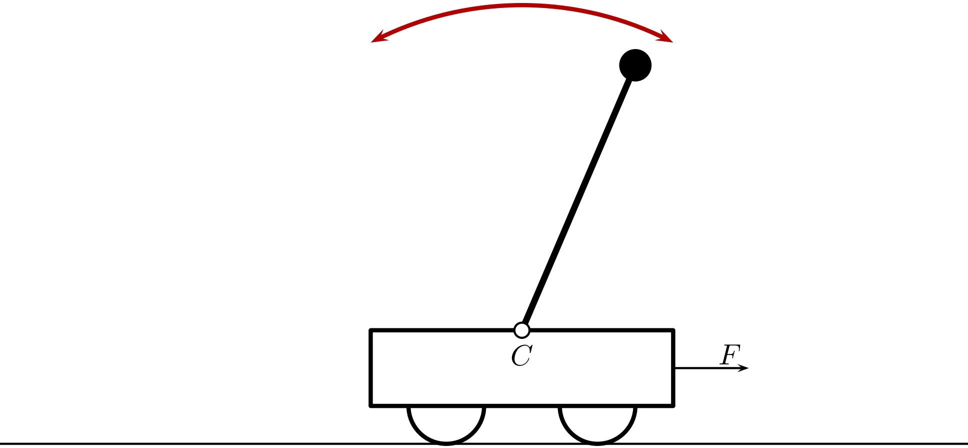
Does there exist $y : [0,T] \times \overline{\Omega} \to \mathbb{R}^n$ and $p : [0,T] \times \overline{\Omega} \to \mathbb{R}$ such that

(2) $$y_t + (y \cdot \nabla)y + \nabla p = 0,\ \text{div } y = 0,$$

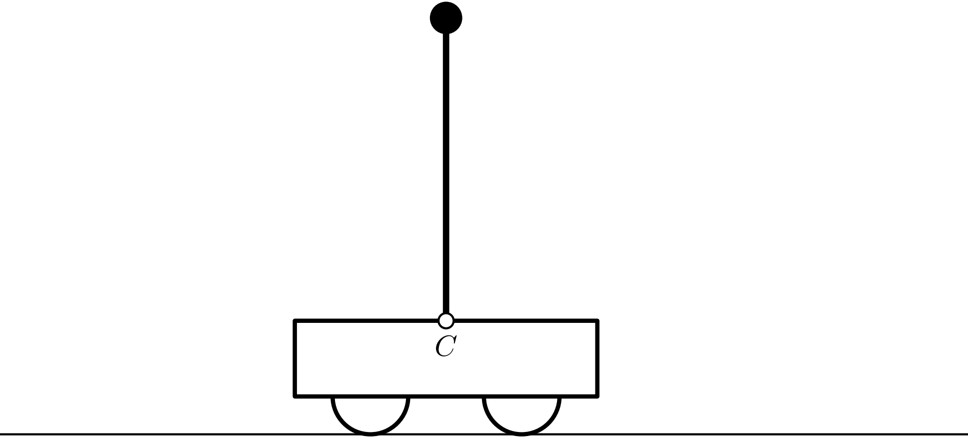
(3) $$y \cdot \nu = 0 \text{ on } [0,T] \times (\partial\Omega \setminus \Gamma),$$

(4) $$y(0,\cdot) = y^0,\ y(T,\cdot) = y^1?$$

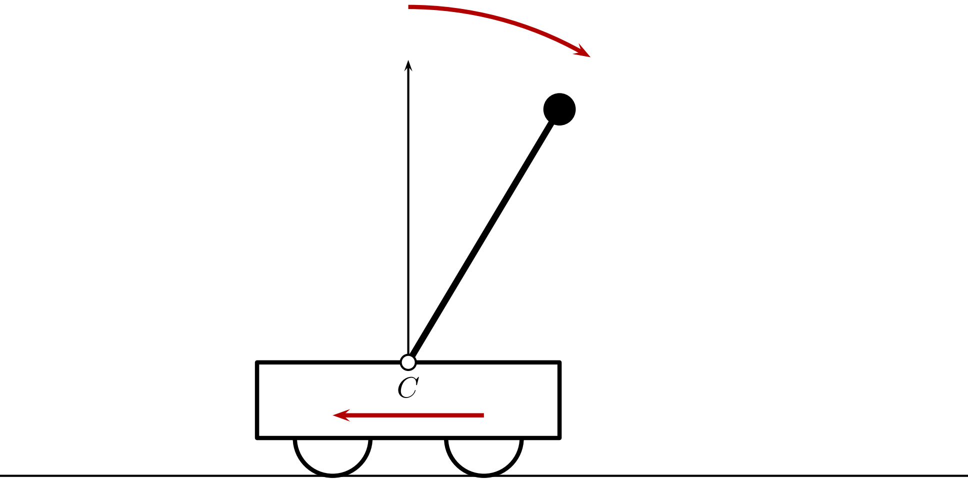
Cart inverted pendulum control system



Cart inverted pendulum: the equilibrium



Instability of the equilibrium



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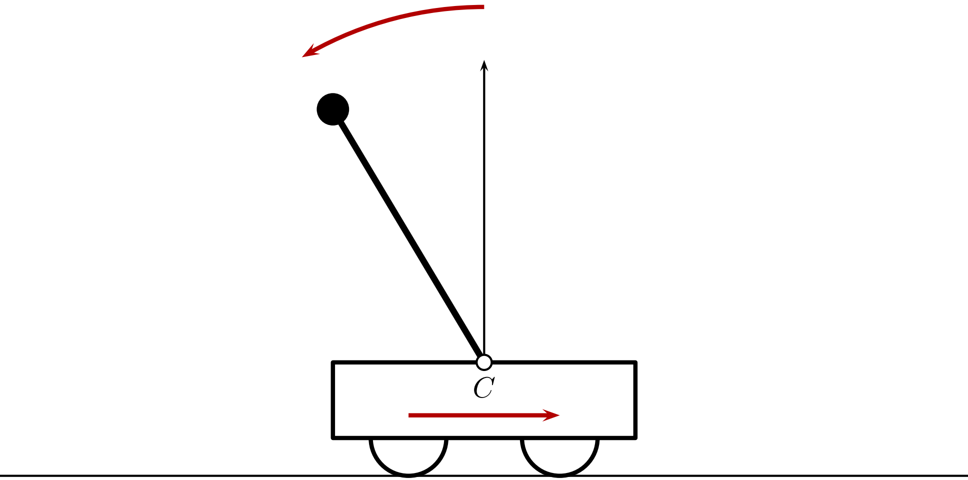
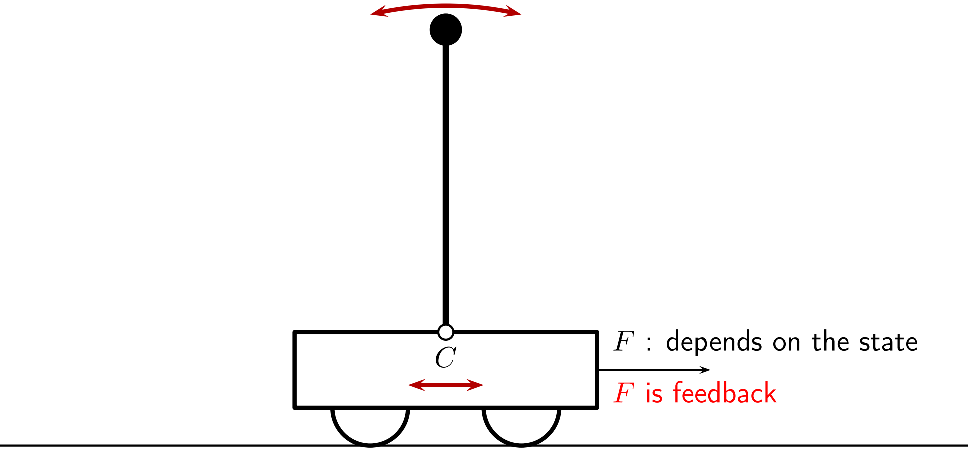
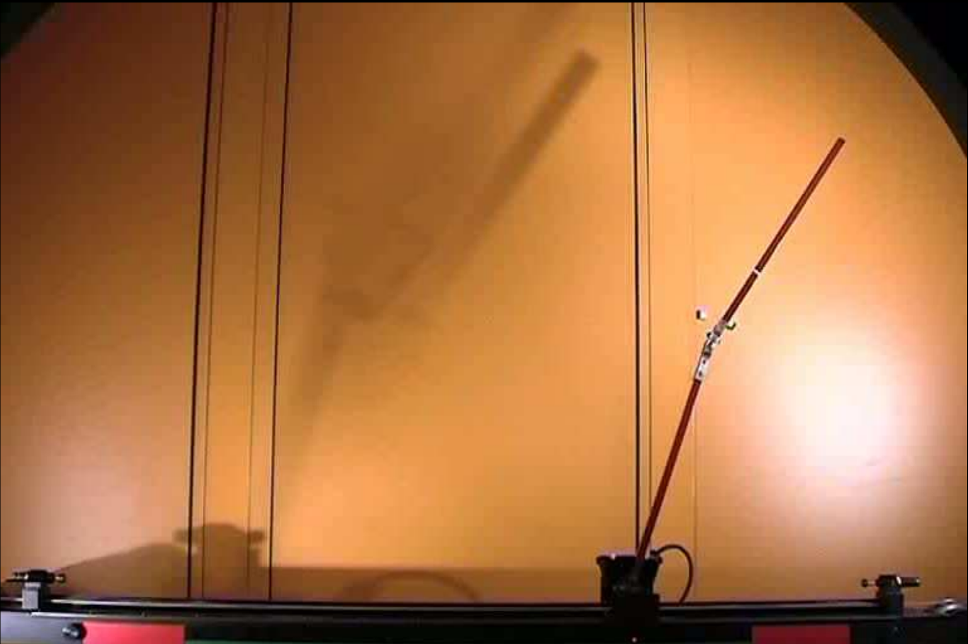


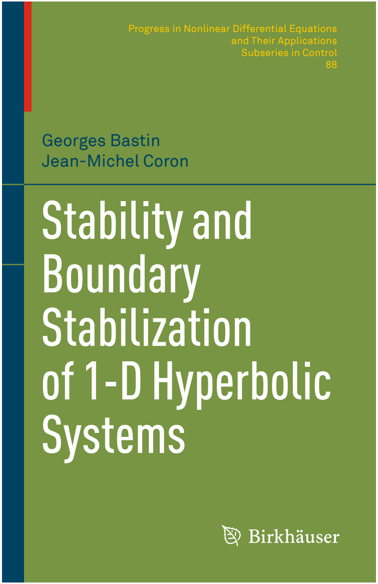
Stabilization of the equilibrium



Double inverted pendulum (CAS, ENSMP/La Villette)



Commercial break (continued)



G. Bastin and JMC, Stability and Boundary Stabilization of 1-D Hyperbolic Systems, 2016, PNLDE Subseries in Control, Birkhaüser.

Controllability of linear control systems

The control system is

$$\dot{y} = Ay + Bu,\ y \in \mathbb{R}^n,\ u \in \mathbb{R}^m,$$

where $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$.

Theorem (Kalman's rank condition)

The linear control system $\dot{y} = Ay + Bu$ is controllable on $[0, T]$ if and only if

$$\textit{Span}\ \{A^i Bu; u \in \mathbb{R}^m,\ \ i \in \{0, 1, \dots, n-1\}\} = \mathbb{R}^n.$$

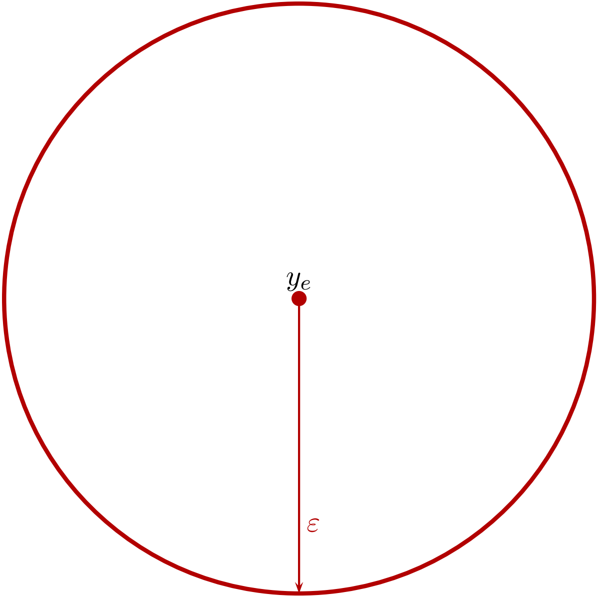
Remark

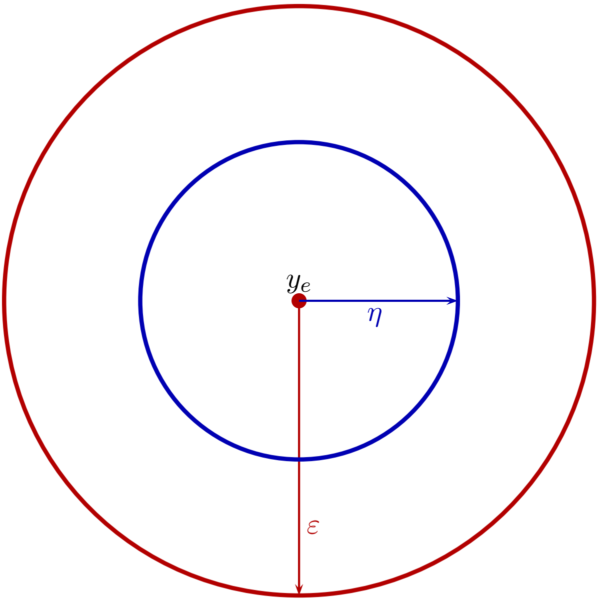
This condition does not depend on T. This is no longer true for nonlinear systems and for systems modeled by linear partial differential equations.

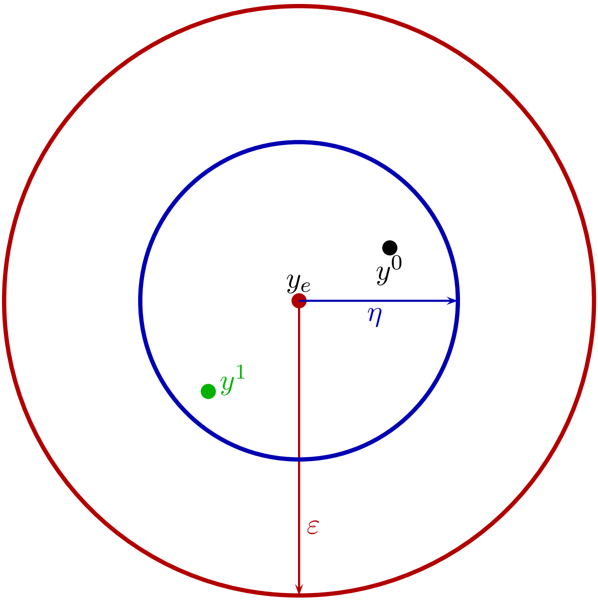
Small time local controllability

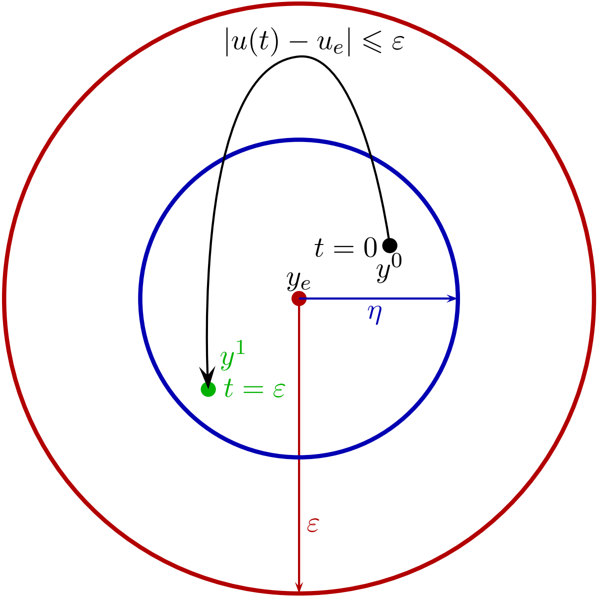
We assume that (y_e, u_e) is an equilibrium, i.e., $f(y_e, u_e) = 0$. Many possible choices for natural definitions of local controllability. The most popular one is **Small-Time Local Controllability (STLC)**: the state remains close to y_e, the control remains to u_e and the time is small.


y_e


y_e
ε


y_e
η
ε


y^0
y_e
η
y^1
ε


$|u(t) - u_e| \leqslant \varepsilon$
$t = 0$
y^0
y_e
η
y^1
$t = \varepsilon$
ε

The linear test

We consider the control system $\dot{y} = f(y, u)$ where the state is $y \in \mathbb{R}^n$ and the control is $u \in \mathbb{R}^m$. Let us assume that $f(y_e, u_e) = 0$. We are interested in the small-time local controllability of $\dot{y} = f(y, u)$ around (y_e, u_e). L. Nirenberg, besides to be a great mathematician, always give great advices when you have no more idea to solve a given problem. I was told that one of his famous advices is

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Have you tried to linearize?

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Have you tried to linearize?

We follow Nirenberg's advice. The linearized control system at (y_e, u_e) is the linear control system $\dot{y} = Ay + Bu$ with

$$A := \frac{\partial f}{\partial y}(y_e, u_e), \; B := \frac{\partial f}{\partial u}(y_e, u_e). \tag{1}$$

If the linearized control system $\dot{y} = Ay + Bu$ is controllable, then $\dot{y} = f(y, u)$ is small-time locally controllable at (y_e, u_e).

Application to the cart-inverted pendulum

For the cart-inverted pendulum, the linearized control system around $(0,0) \in \mathbb{R}^4 \times \mathbb{R}$ is $\dot{y} = Ay + Bu$ with

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -\frac{mg}{M} & 0 & 0 \\ 0 & \frac{(M+m)g}{Ml} & 0 & 0 \end{pmatrix}, \quad B = \frac{1}{Ml} \begin{pmatrix} 0 \\ 0 \\ l \\ -1 \end{pmatrix}. \tag{1}$$

One easily checks that this linearized control system satisfies the Kalman rank condition and therefore is controllable. Hence the cart-inverted pendulum is small-time locally controllable at $(0,0) \in \mathbb{R}^4 \times \mathbb{R}$.

The linear test and the controllability of the baby stroller

Let us recall that the baby stroller control system is

(1) $$\dot{y}_1 = u_1 \cos y_3,\ \dot{y}_2 = u_1 \sin y_3, \dot{y}_3 = u_2,\ n = 3,\ m = 2.$$

The linearized control system at $(0,0) \in \mathbb{R}^3 \times \mathbb{R}^2$ is

(2) $$\dot{y}_1 = u_1,\ \dot{y}_2 = 0,\ \dot{y}_3 = u_2,$$

which is clearly not controllable. The linearized control system gives no information on the small-time local controllability at $(0,0) \in \mathbb{R}^3 \times \mathbb{R}^2$ of the baby stroller.

What to do if linearized control system is not controllable?

Question: What to do if

$$\dot{y} = \frac{\partial f}{\partial y}(y_e, u_e)y + \frac{\partial f}{\partial u}(y_e, u_e)u \tag{1}$$

is not controllable?

In finite dimension: One uses **iterated Lie brackets**.

Lie brackets and iterated Lie brackets

Definition (Lie brackets)

(1) $$[X,Y](y) := Y'(y)X(y) - X'(y)Y(y).$$

Iterated Lie brackets: $[X,[X,Y]]$, $[[Y,X],[X,[X,Y]]]$ etc. For simplicity, from now on we assume that

(2) $$f(y,u) = f_0(y) + \sum_{i=1}^{m} u_i f_i(y) \text{ with } f_0(0) = 0.$$

Drift: f_0. **Driftless control systems:** $f_0 = 0$. We denote by Lie $\{f_0, f_1, \ldots, f_m\}$ the smallest vector subspace $\mathcal{E}$ of $C^\infty(\mathbb{R}^n;\mathbb{R}^n)$ containing f_0, f_1,..., f_m which is stable for the Lie bracket: if $X \in \mathcal{E}$ and $Y \in \mathcal{E}$, then $[X,Y] \in \mathcal{E}$.

Lie bracket for $\dot{y} = u_1 f_1(y) + u_2 f_2(y)$

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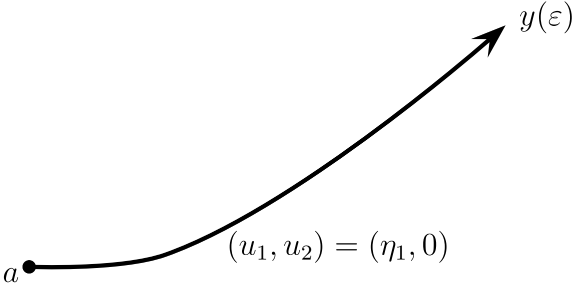

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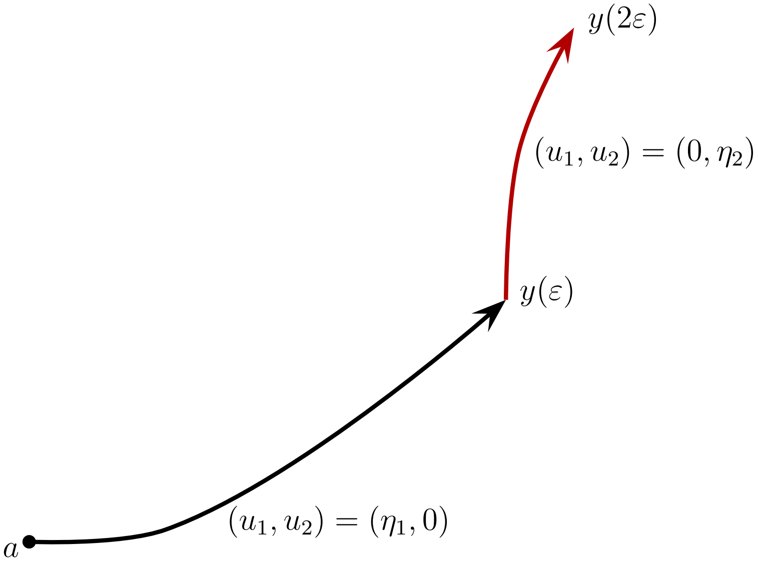

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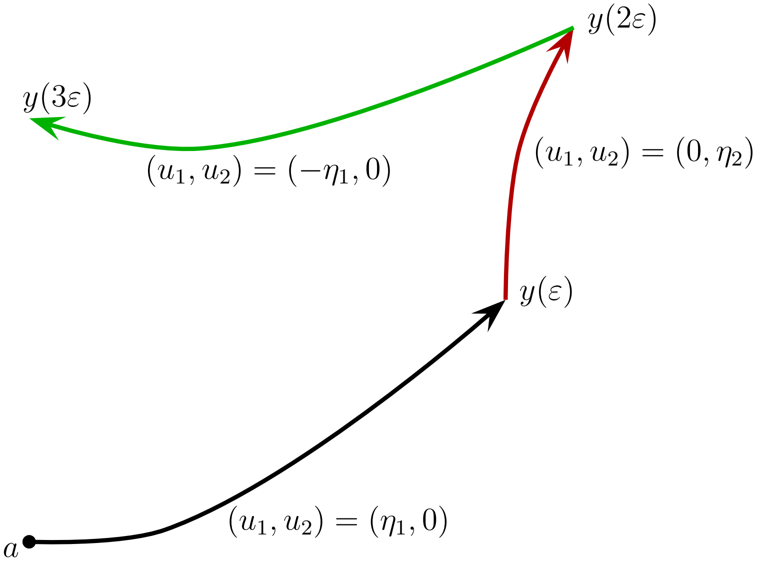

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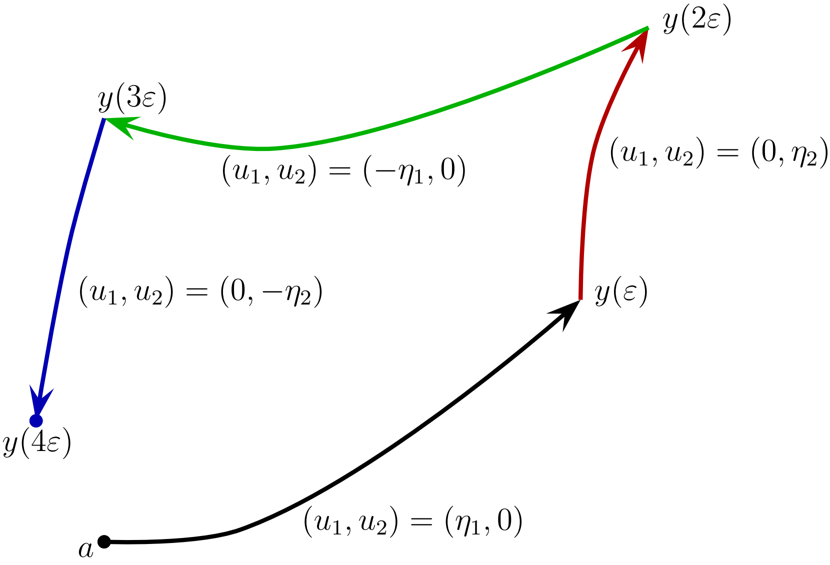

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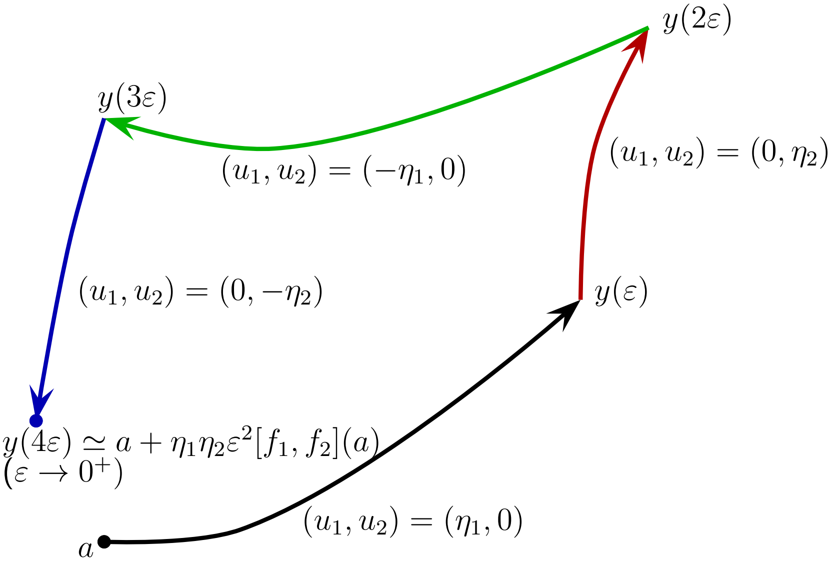


Controllability of driftless control systems: Local controllability

Theorem (P. Rashevski (1938), W.-L. Chow (1939))

Let $\mathcal{O}$ be a nonempty open subset of $\mathbb{R}^n$ and let $y_e \in \mathcal{O}$. Let us assume that, for some $f_1, \ldots, f_m : \mathcal{O} \rightarrow \mathbb{R}^n$,

$$(1) \qquad f(y,u) = \sum_{i=1}^{m} u_i f_i(y), \ \forall (y,u) \in \mathcal{O} \times \mathbb{R}^m.$$

Let us also assume that

$$(2) \qquad \left\{ h(y_e);\ h \in \text{Lie}\ \{f_1, \ldots, f_m\} \right\} = \mathbb{R}^n.$$

Then the control system $\dot{y} = f(y,u)$ is small-time locally controllable at $(y_e, 0) \in \mathbb{R}^n \times \mathbb{R}^m$.

The baby stroller system: Controllability

(1) $$\dot{y}_1 = u_1 \cos y_3,\ \dot{y}_2 = u_1 \sin y_3, \dot{y}_3 = u_2,\ n = 3,\ m = 2.$$

This system can be written as $\dot{y} = u_1 f_1(y) + u_2 f_2(y)$, with

(2) $$f_1(y) = (\cos y_3, \sin y_3, 0),\ f_2(y) = (0, 0, 1).$$

One has

(3) $$[f_1, f_2](y) = (\sin y_3, -\cos y_3, 0).$$

Hence $f_1(0)$, $f_2(0)$ and $[f_1, f_2](0)$ all together span all of $\mathbb{R}^3$. This implies the small-time local controllability of the baby stroller at $(0, 0) \in \mathbb{R}^3 \times \mathbb{R}^2$.

Drift and the Lie algebra rank condition

We consider the control affine system $\dot{y} = f_0(y) + \sum_{i=1}^{m} u_i f_i(y)$ with $f_0(0) = 0$. One says that this control system satisfies the **Lie algebra rank condition** at $0 \in \mathbb{R}^n$ if

(1) $$\left\{h(0);\ h \in \text{Lie}\ \{f_0, f_1, \dots, f_m\}\right\} = \mathbb{R}^n.$$

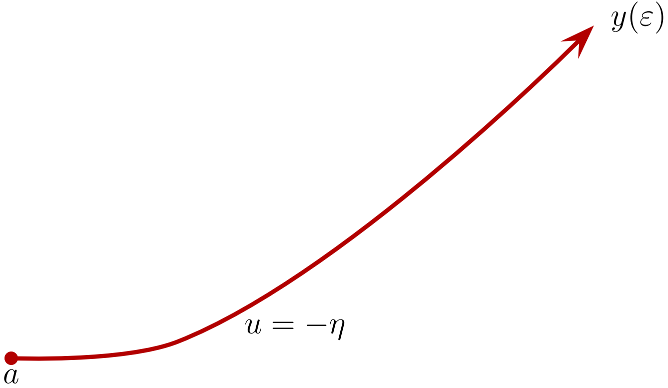
One has the following theorem.

Theorem (R. Hermann (1963) and T. Nagano (1966))

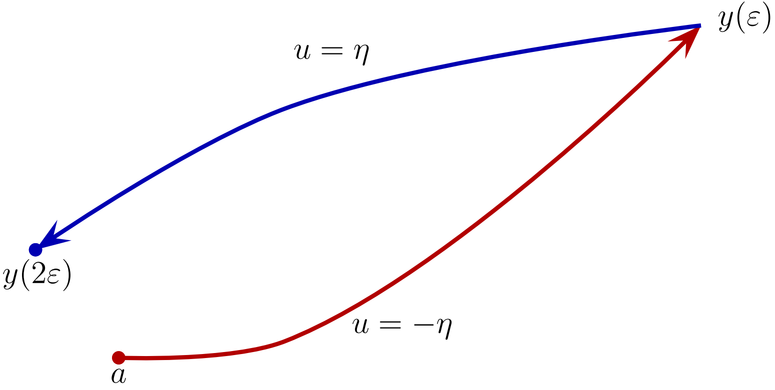
If the f_i's are analytic in a neighborhood of $0 \in \mathbb{R}^n$ and if the control system $\dot{y} = f_0(y) + \sum_{i=1}^{m} f_i(y)$ is small-time locally controllable at $(0,0) \in \mathbb{R}^n \times \mathbb{R}^m$, then this control system satisfies the Lie algebra rank condition at $0 \in \mathbb{R}^n$.

Lie bracket for $\dot{y} = f_0(y) + uf_1(y)$, with $f_0(a) = 0$

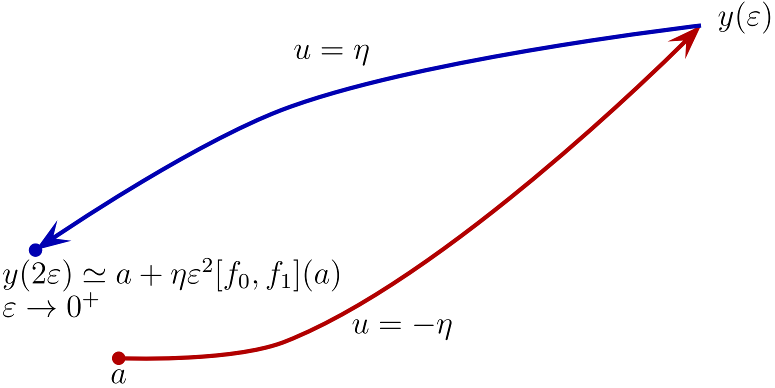
Lie bracket for $\dot{y} = f_0(y) + uf_1(y)$, with $f_0(a) = 0$



Lie bracket for $\dot{y} = f_0(y) + uf_1(y)$, with $f_0(a) = 0$



Lie bracket for $\dot{y} = f_0(y) + u f_1(y)$, with $f_0(a) = 0$



The Kalman rank condition and the Lie algebra rank condition

Let us write the linear control system $\dot{y} = Ay + Bu$ as $\dot{y} = f_0(y) + \sum_{i=1}^{m} u_i f_i(y)$ with

(1) $$f_0(y) = Ay,\ f_i(y) = B_i,\ B_i \in \mathbb{R}^n,\ (B_1, \ldots, B_m) = B.$$

The Kalman rank condition is equivalent to the Lie algebra rank condition at $0 \in \mathbb{R}^n$.
Hence the Lie algebra rank condition is sufficient for two important cases, namely Linear systems and driftless control systems.

With a drift term: Not all the iterated Lie brackets are good

We take $n = 2$ and $m = 1$ and consider the control system

(1) $$\dot{y}_1 = y_2^2,\ \dot{y}_2 = u,$$

where the state is $y := (y_1, y_2) \in \mathbb{R}^2$ and the control is $u \in \mathbb{R}$. This control system can be written as $\dot{y} = f_0(y) + uf_1(y)$ with

(2) $$f_0(y) = (y_2^2, 0), f_1(y) = (0, 1).$$

One has $[f_1, [f_1, f_0]] = (2, 0)$ and therefore $f_1(0)$ and $[f_1, [f_1, f_0]](0)$ span all of $\mathbb{R}^2$. However the control system (1) is clearly not small-time locally controllable at $(0, 0) \in \mathbb{R}^2 \times \mathbb{R}$.

References for sufficient or necessary conditions for small-time local controllability when there is a drift term

- A. Agrachev (1991),
- A. Agrachev and R. Gamkrelidze (1993),
- R. M. Bianchini and Stefani (1986),
- H. Frankowska (1987),
- M. Kawski (1990),
- H. Sussmann (1983, 1987),
- A. Tret'yak (1990),
- K. Beauchard and F. Marbach (2017).

The under-actuated satellite

(1) $$\dot{\omega} = J^{-1}S(\omega)J\omega + \sum_{i=1}^{m} u_i J^{-1} b_i,\ \dot{\eta} = A(\eta)\omega,$$

with $S(\omega)x := x \wedge \omega$. One has $A(0) = \text{Id}$. The vectors $b_1, \ldots, b_m$ are independent. If $m = 3$, the linearized control system around the equilibrium $(0,0) \in \mathbb{R}^6 \times \mathbb{R}^3$ is controllable and the control system is small-time locally controllable at $(0,0) \in \mathbb{R}^6 \times \mathbb{R}^3$. We now turn to the case where $m = 2$. One easily sees that the linearized control system around the equilibrium $(0,0) \in \mathbb{R}^6 \times \mathbb{R}^2$ is not controllable. However, if

(2) $$\text{Span } \{b_1, b_2, S(\omega)J^{-1}\omega;\, \omega \in \text{ Span } \{b_1, b_2\}\} = \mathbb{R}^3,$$

then the control system (1) is small-time locally controllable at $(0,0) \in \mathbb{R}^6 \times \mathbb{R}^2$. (This follows from a sufficient condition for local controllability proved by H. Sussmann in 1987.)

Iterated Lie brackets and PDE control systems

Iterative Lie brackets have been used successfully for some control PDE systems:

- Euler and Navier Stokes control systems (different from the one considered here): A. Agrachev and A. Sarychev (2005); A. Shirikyan (2006, 2007), H. Nersisyan (2010),
- Schrödinger control system: T. Chambrion, P. Mason, M. Sigalotti and U. Boscain (2009), U. Boscain, F. Chittaro, P. Mason, M. Sigalotti (2012), U. Boscain, M. Caponigro, T. Chambrion, and M. Sigalotti (2012).

However, for many PDE, one does not know how to use them.

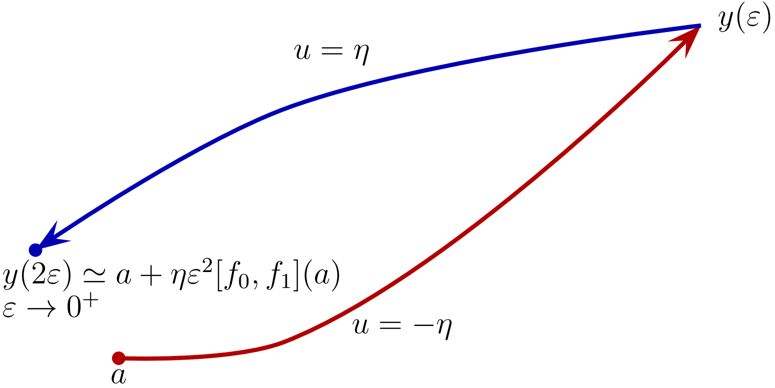
A problem with Lie brackets for PDE control systems

Consider the simplest PDE control system

$$y_t + y_x = 0,\ x \in [0, L],\ y(t, 0) = u(t). \tag{1}$$

It is a control system where, at time t, the state is $y(t, \cdot) : (0, L) \to \mathbb{R}$ and the control is $u(t) \in \mathbb{R}$. Formally it can be written in the form $\dot{y} = f_0(y) + uf_1(y)$. Here f_0 is linear and f_1 is constant.

Lie bracket for $\dot{y} = f_0(y) + uf_1(y)$, with $f_0(a) = 0$



Problems of the Lie brackets for PDE control systems (continued)

Let us consider, for $\varepsilon > 0$, the control defined on $[0, 2\varepsilon]$ by

$$u(t) := -\eta \text{ for } t \in (0, \varepsilon),\ u(t) := \eta \text{ for } t \in (\varepsilon, 2\varepsilon). \tag{1}$$

Let $y : (0, 2\varepsilon) \times (0, L) \to \mathbb{R}$ be the solution of the Cauchy problem

$$y_t + y_x = 0,\ t \in (0, 2\varepsilon),\ x \in (0, L), \tag{2}$$

$$y(t, 0) = u(t),\ t \in (0, 2\varepsilon),\ y(0, x) = 0,\ x \in (0, L). \tag{3}$$

Then one readily gets, if $2\varepsilon \leqslant L$,

$$y(2\varepsilon, x) = \eta,\ x \in (0, \varepsilon),\ y(2\varepsilon, x) = -\eta,\ x \in (\varepsilon, 2\varepsilon), \tag{4}$$

$$y(2\varepsilon, x) = 0,\ x \in (2\varepsilon, L). \tag{5}$$

Problems of the Lie brackets for PDE control systems (continued)

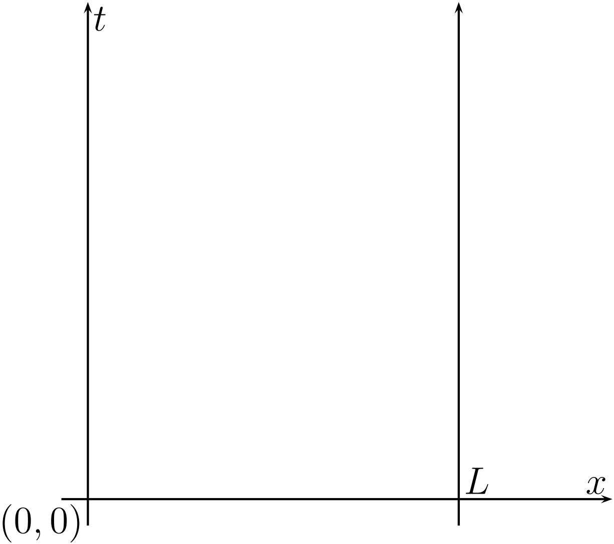
(1) $$\left|\frac{y(2\varepsilon,\cdot)-y(0,\cdot)}{\varepsilon^2}\right|_{L^2(0,L)} \to +\infty \text{ as } \varepsilon \to 0^+.$$

For every $\phi \in H^2(0,L)$, one gets after suitable computations

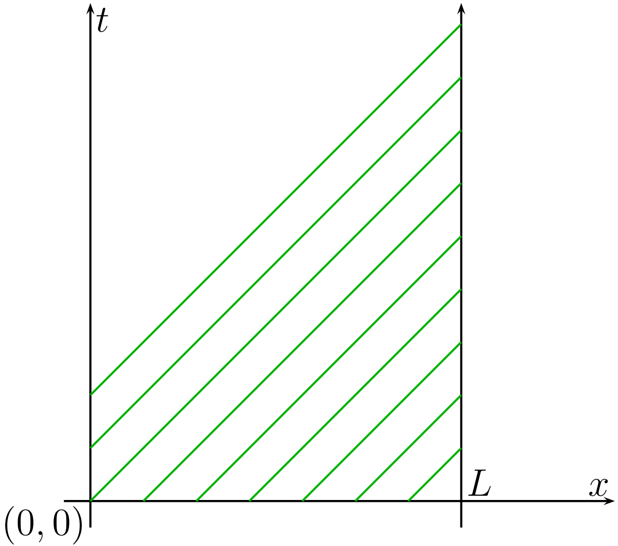
(2) $$\lim_{\varepsilon\to 0^+}\int_0^L \phi(x)\left(\frac{y(2\varepsilon,x)-y(0,x)}{\varepsilon^2}\right)dx = -\eta\phi'(0).$$

So, in some sense, we could say that $[f_0, f_1] = \delta_0'$. Unfortunately it is not clear how to use this derivative of a Dirac mass at 0.

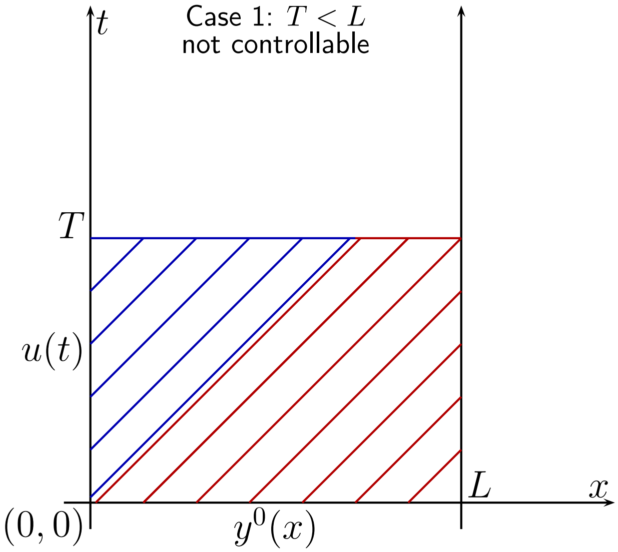
Controllability of $y_t + y_x = 0$, $x \in [0, L]$, $y(t, 0) = u(t)$



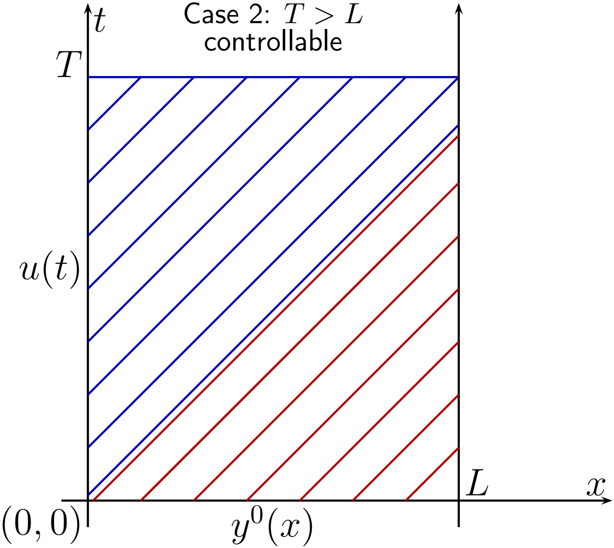
Controllability of $y_t + y_x = 0$, $x \in [0, L]$, $y(t, 0) = u(t)$



Controllability of $y_t + y_x = 0$, $x \in [0, L]$, $y(t,0) = u(t)$



Controllability of $y_t + y_x = 0$, $x \in [0, L]$, $y(t, 0) = u(t)$



Controllability of control systems modeled by linear PDE

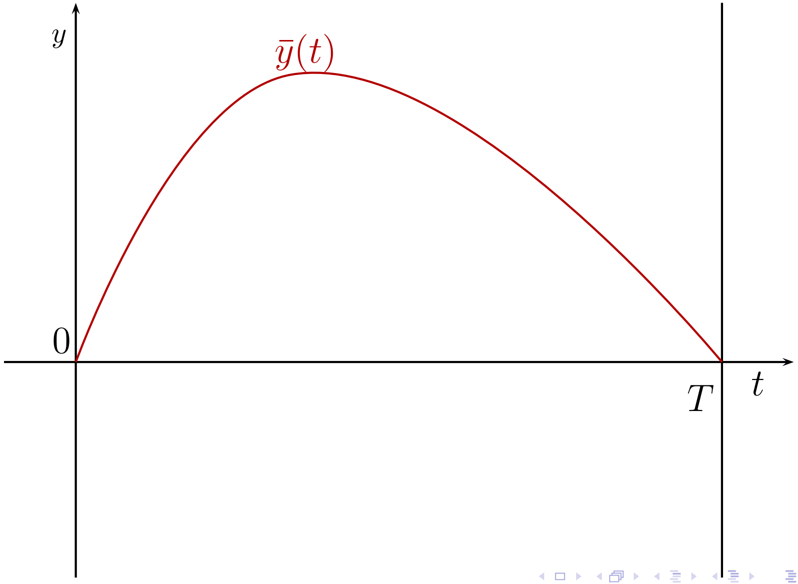
There are lot of powerful tools to study the controllability of linear control systems in infinite dimension. The most popular ones are based on the duality between observability and controllability. This leads to try to prove **observability inequalities**. There are many methods to prove this observability inequalities. For example:

- Ingham's inequalities and harmonic analysis: D. Russell (1967),
- Multipliers method: Lop Fat Ho (1986), J.-L. Lions (1988),
- Microlocal analysis: C. Bardos-G. Lebeau-J. Rauch (1992),
- Carleman's inequalities: A. Fursikov, O. Imanuvilov, G. Lebeau and L. Robbiano (1993-1996).

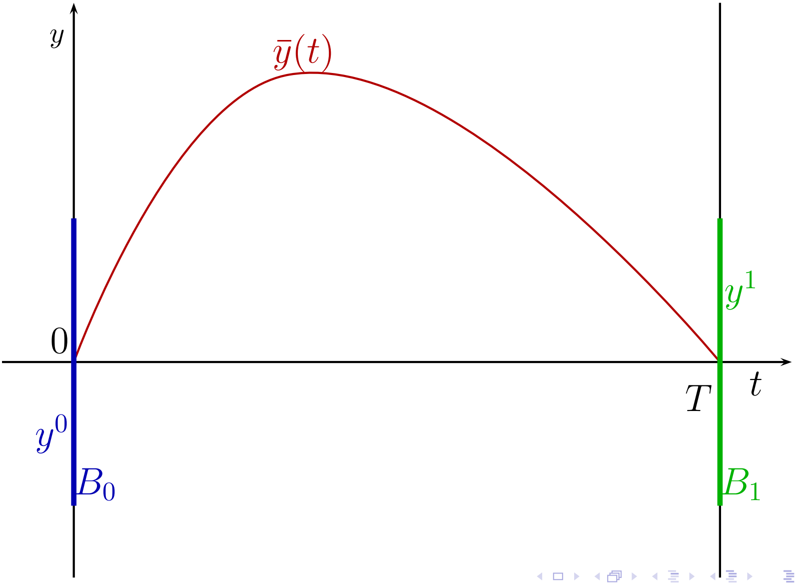
(However there are still plenty of open problems.) What to do if the linearized control system around the equilibrium of interest is not controllable?

A method to avoid Lie brackets: The return method (JMC (1992))

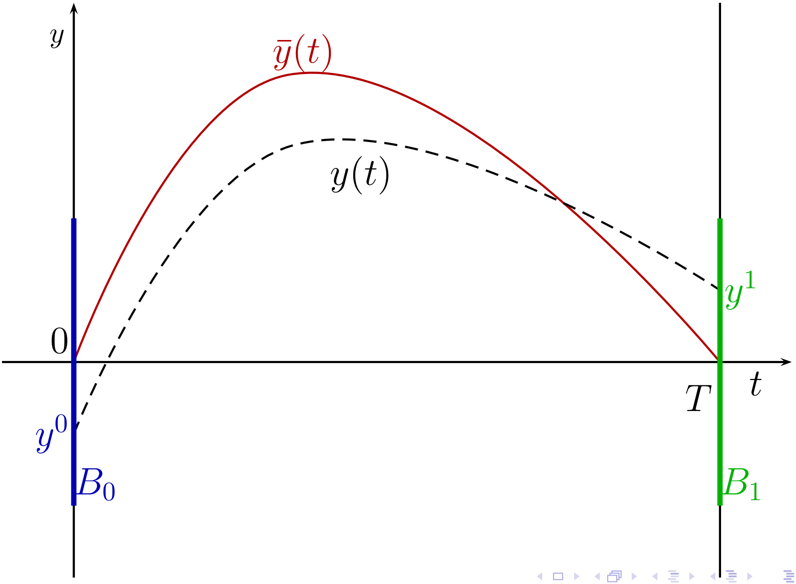
A method to avoid Lie brackets: The return method (JMC (1992))



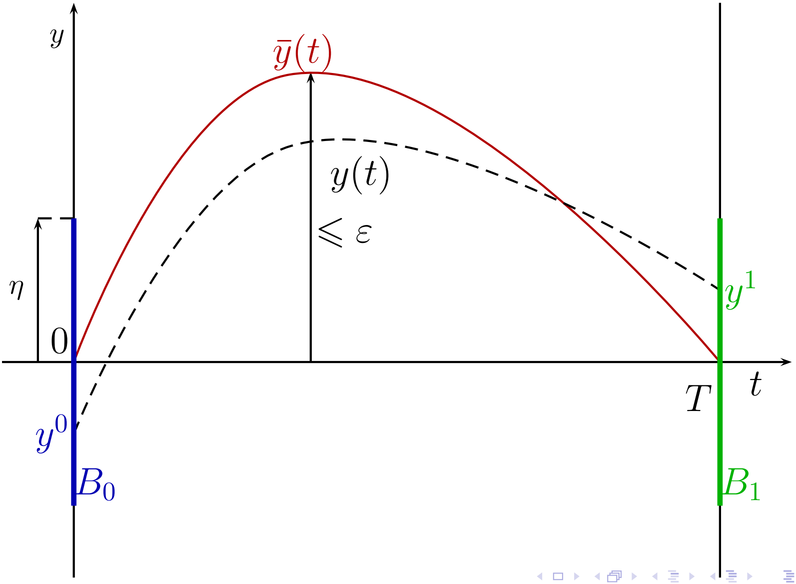
A method to avoid Lie brackets: The return method (JMC (1992))



A method to avoid Lie brackets: The return method (JMC (1992))



A method to avoid Lie brackets: The return method (JMC (1992))



The return method: An example in finite dimension

We go back to the baby stroller control system

$$\dot{y}_1 = u_1 \cos y_3,\ \dot{y}_2 = u_1 \sin y_3,\ \dot{y}_3 = u_2. \tag{1}$$

For every $\bar{u} : [0,T] \to \mathbb{R}^2$ such that, for every t in $[0,T]$, $\bar{u}(T-t) = -\bar{u}(t)$, every solution $\bar{y} : [0,T] \to \mathbb{R}^3$ of

$$\dot{\bar{y}}_1 = \bar{u}_1 \cos \bar{y}_3,\ \dot{\bar{y}}_2 = \bar{u}_1 \sin \bar{y}_3, \dot{\bar{y}}_3 = \bar{u}_2, \tag{2}$$

satisfies $\bar{y}(0) = \bar{y}(T)$. The linearized control system around $(\bar{y}, \bar{u})$ is

$$\begin{cases} \dot{y}_1 = -\bar{u}_1 y_3 \sin \bar{y}_3 + u_1 \cos \bar{y}_3,\ \dot{y}_2 = \bar{u}_1 y_3 \cos \bar{y}_3 + u_1 \sin \bar{y}_3, \\ \dot{y}_3 = u_2, \end{cases} \tag{3}$$

which is controllable if (and only if) $\bar{u} \not\equiv 0$. We have got the controllability of the baby stroller system without using Lie brackets. We have only used controllability results for **linear** control systems.

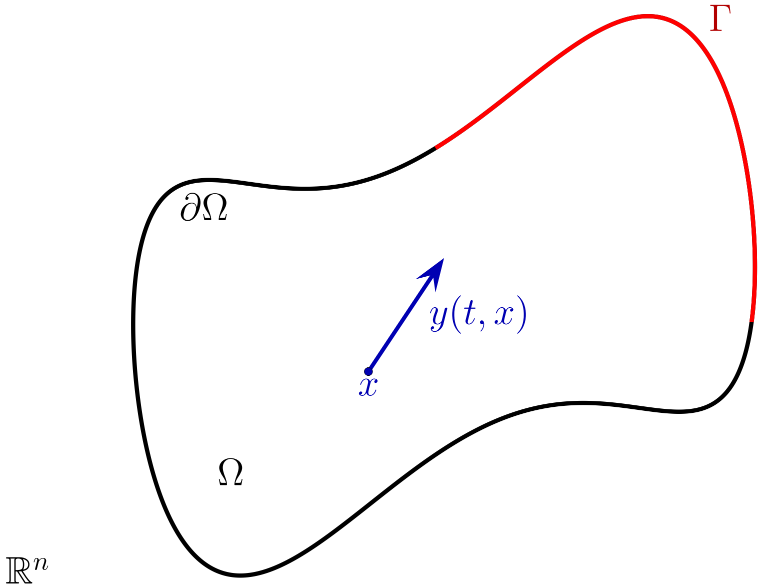
Return method: References

- Stabilization of driftless systems in finite dimension: JMC (1992),
- Euler equations of incompressible fluids: JMC (1993,1996), O. Glass (1997,2000), O. Glass-Th. Horsin (2010, 2012, 2016),
- Control of driftless systems in finite dimension: E.D. Sontag (1995),
- Navier-Stokes equations: JMC (1996), JMC and A. Fursikov (1996), A. Fursikov and O. Imanuvilov (1999), S. Guerrero, O. Imanuvilov and J.-P. Puel (2006), JMC and S. Guerrero (2009), M. Chapouly (2009), JMC and P. Lissy (2014),
- Saint-Venant equations: JMC (2002),
- Vlasov Poisson: O. Glass (2003),

Return method: References (continued)

- Isentropic Euler equations: O. Glass (2006),
- Schrödinger equation: K. Beauchard (2005), K. Beauchard and JMC (2006),
- Hyperbolic/wave equations: JMC, O. Glass and Z. Wang (2009), F. Alabau, JMC and G. Olive (2017), C. Zhang (2017),
- Ensemble controllability of Bloch equations: K. Beauchard, JMC and P. Rouchon (2010),
- Parabolic systems: JMC, S. Guerrero and L. Rosier (2010), JMC and J.-Ph. Guilleron (2017),
- Uniform controllability of scalar conservation laws in the vanishing viscosity limit: M. Léautaud (2010).

The Euler control system



Controllability problem

We denote by $\nu : \partial\Omega \to \mathbb{R}^n$ the outward unit normal vector field to Ω. Let $T > 0$. Let y^0, $y^1 : \overline{\Omega} \to \mathbb{R}^n$ be such that

(1) $$\text{div } y^0 = \text{div } y^1 = 0,\ y^0 \cdot \nu = y^1 \cdot \nu = 0 \text{ on } \partial\Omega \setminus \Gamma.$$

Does there exist $y : [0,T] \times \overline{\Omega} \to \mathbb{R}^n$ and $p : [0,T] \times \overline{\Omega} \to \mathbb{R}$ such that

(2) $$y_t + (y \cdot \nabla)y + \nabla p = 0,\ \text{div } y = 0,$$

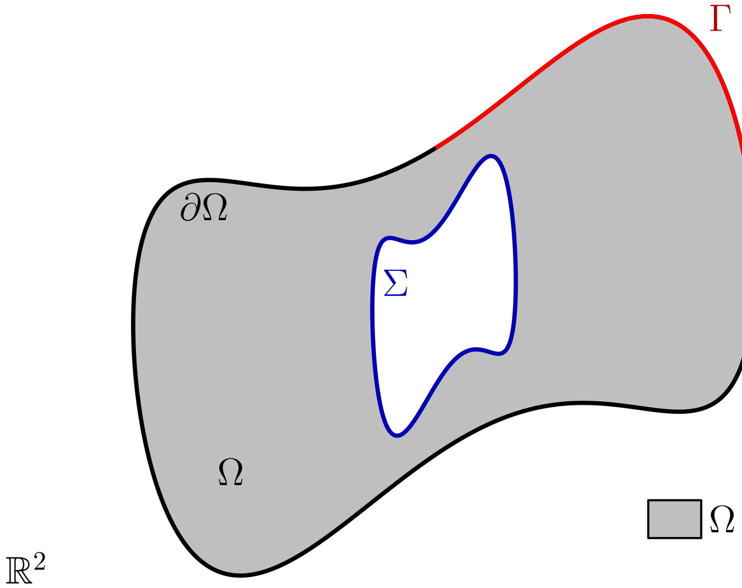
(3) $$y \cdot \nu = 0 \text{ on } [0,T] \times (\partial\Omega \setminus \Gamma),$$

(4) $$y(0,\cdot) = y^0,\ y(T,\cdot) = y^1?$$

For the control, many choices are in fact possible. For example, for $n = 2$, one can take

1. $y \cdot \nu$ on Γ with $\int_\Gamma y \cdot \nu = 0$,
2. $\mathrm{curl}\ y$ at the points of $[0,T] \times \Gamma$ where $y \cdot \nu < 0$.

A case without controllability



Proof of the noncontrollability

Let us give it only for $n = 2$. Let γ_0 be a Jordan curve in $\overline{\Omega}$. Let, for $t \in [0, T]$, $\gamma(t)$ be the Jordan curve obtained, at time $t \in [0, T]$, from the points of the fluids which, at time 0, were on γ_0. The **Kelvin law** tells us that, if $\gamma(t)$ does not intersect Γ,

(1) $$\int_{\gamma(t)} y(t, \cdot) \cdot \overrightarrow{ds} = \int_{\gamma_0} y(0, \cdot) \cdot \overrightarrow{ds}, \, \forall t \in [0, T],$$

We take $\gamma_0 := \Sigma$. Then $\gamma(t) = \Sigma$ for every $t \in [0, T]$. Hence, if

(2) $$\int_{\Sigma} y^1 \cdot \overrightarrow{ds} \neq \int_{\Sigma} y^0 \cdot \overrightarrow{ds},$$

one cannot steer the control system from y^0 to y^1.
More generally, for every $n \in \{2, 3\}$, if Γ does not intersect every connected component of the boundary $\partial\Omega$ of Ω, the Euler control system is not controllable. This is the only obstruction to the controllability of the Euler control system.

Controllability of the Euler control system

Theorem (JMC for $n = 2$ (1996), O. Glass for $n = 3$ (2000))

Assume that Γ intersects every connected component of $\partial\Omega$. Then the Euler control system is globally controllable in every time: For every $T > 0$, for every y^0, $y^1 : \overline{\Omega} \to \mathbb{R}^n$ such that

$$\text{div } y^0 = \text{div } y^1 = 0,\ y^0 \cdot \nu = y^1 \cdot \nu = 0 \text{ on } \partial\Omega \setminus \Gamma, \tag{1}$$

there exist $y : [0, T] \times \overline{\Omega} \to \mathbb{R}^n$ and $p : [0, T] \times \overline{\Omega} \to \mathbb{R}$ such that

$$y_t + (y \cdot \nabla)y + \nabla p = 0,\ \text{div } y = 0, \tag{2}$$

$$y \cdot \nu = 0 \text{ on } [0, T] \times (\partial\Omega \setminus \Gamma) \tag{3}$$

$$y(0, \cdot) = y^0,\ y(T, \cdot) = y^1. \tag{4}$$

Comments on the proof

First try: One studies the controllability of the linearized control system around 0. This linearized control system is the control system

$$(1) \qquad y_t + \nabla p = 0, \text{ div } y = 0, \; y \cdot \nu = 0 \text{ on } [0, T] \times (\partial\Omega \setminus \Gamma).$$

For simplicity we assume that $n = 2$. Taking the curl of the first equation, on gets,

$$(2) \qquad (\text{curl } y)_t = 0.$$

Hence curl y remains constant along the trajectories of the linearized control system. Hence **the linearized control system is not controllable**.

The return method and the controllability of the Euler equations

One looks for $(\bar{y}, \bar{p}) : [0, T] \times \overline{\Omega} \to \mathbb{R}^n \times \mathbb{R}$ such that

$$\bar{y}_t + (\bar{y} \cdot \nabla \bar{y}) + \nabla \bar{p} = 0, \text{ div } \bar{y} = 0, \tag{1}$$

$$\bar{y} \cdot \nu = 0 \text{ on } [0, T] \times (\partial\Omega \setminus \Gamma), \tag{2}$$

$$\bar{y}(T, \cdot) = \bar{y}(0, \cdot) = 0, \tag{3}$$

$$\text{the linearized control system around } (\bar{y}, \bar{p}) \text{ is controllable.} \tag{4}$$

Construction of $(\bar{y}, \bar{p})$

Take $\theta : \overline{\Omega} \to \mathbb{R}$ such that

$$\Delta\theta = 0 \text{ in } \Omega,\ \frac{\partial\theta}{\partial\nu} = 0 \text{ on } \partial\Omega \setminus \Gamma.$$

Take $\alpha : [0, T] \to \mathbb{R}$ such that $\alpha(0) = \alpha(T) = 0$. Finally, define $(\bar{y}, \bar{p}) : [0, T] \times \overline{\Omega} \to \mathbb{R}^2 \times \mathbb{R}$ by

(1) $$\bar{y}(t, x) := \alpha(t)\nabla\theta(x),\ \bar{p}(t, x) := -\dot{\alpha}(t)\theta(x) - \frac{\alpha(t)^2}{2}|\nabla\theta(x)|^2.$$

Then $(\bar{y}, \bar{p})$ is a trajectory of the Euler control system which goes from 0 to 0.

Controllability of the linearized control system around $(\bar{y}, \bar{p})$ if $n = 2$

The linearized control system around $(\bar{y}, \bar{p})$ is

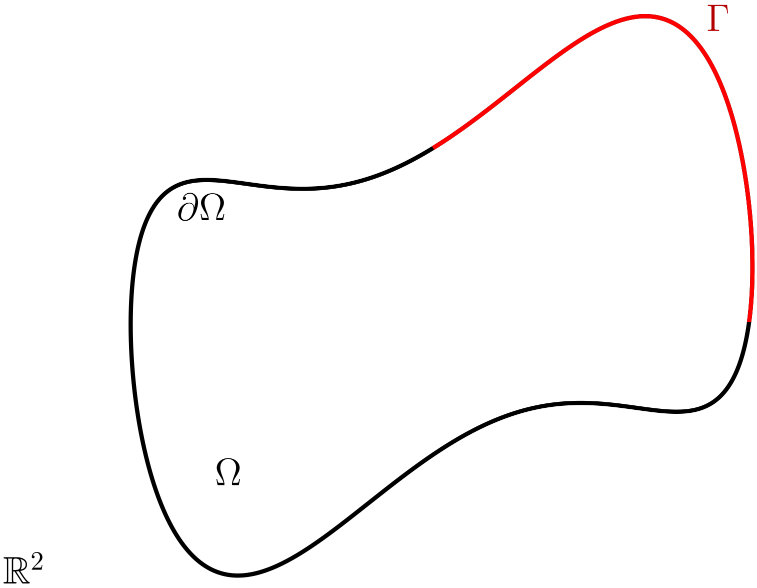
$$(1)\begin{cases} y_t + (\bar{y} \cdot \nabla)y + (y \cdot \nabla)\bar{y} + \nabla p = 0, \ \text{div } y = 0 \text{ in } [0, T] \times \Omega, \\ y \cdot \nu = 0 \text{ on } [0, T] \times (\partial\Omega \setminus \Gamma). \end{cases}$$

Again we assume that $n = 2$. Taking once more the curl of the first equation, one gets

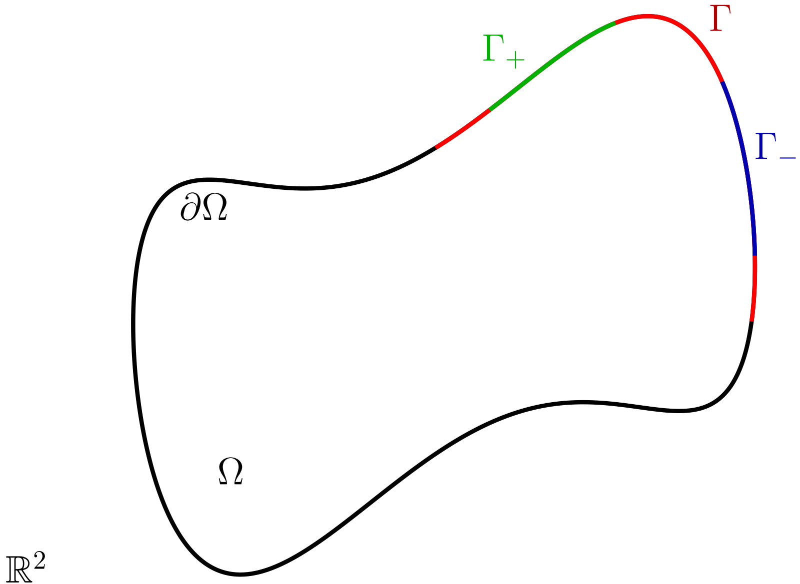
$$(2) \qquad (\text{curl } y)_t + (\bar{y} \cdot \nabla)(\text{curl } y) = 0.$$

This is a simple transport equation on curl y. If there exists $a \in \overline{\Omega}$ such that $\nabla\theta(a) = 0$, then $\bar{y}(t, a) = 0$ and $(\text{curl } y)_t(t, a) = 0$ showing that (2) is not controllable. This is the only obstruction: If $\nabla\theta$ does not vanish in $\overline{\Omega}$, one can prove that (2) (and then (1)) is controllable if $\int_0^T \alpha(t)dt$ is large enough.

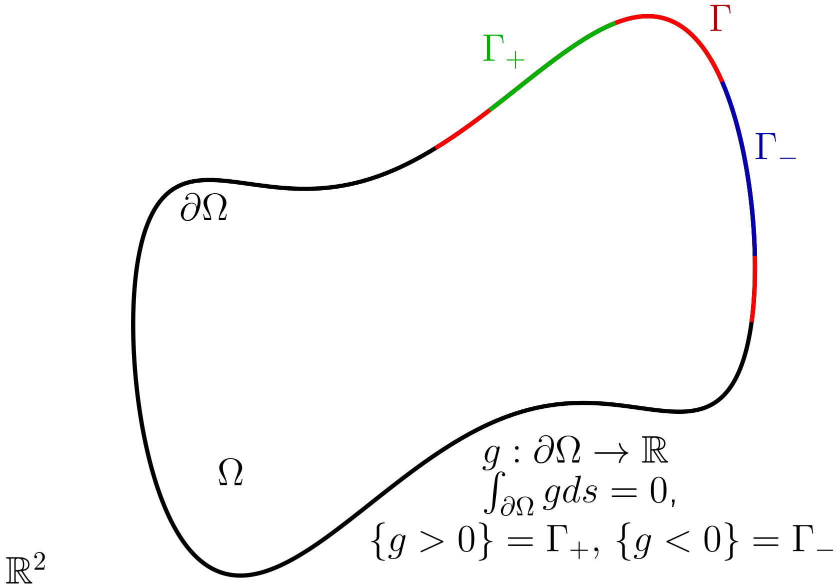
A good θ for $n = 2$ and Ω simply connected



A good θ for $n = 2$ and Ω simply connected



A good θ for $n = 2$ and Ω simply connected

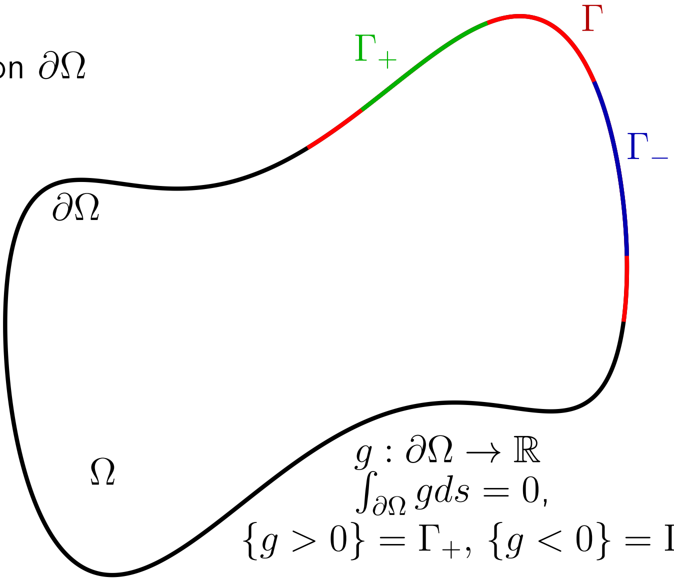


$$g : \partial\Omega \to \mathbb{R}$$
$$\int_{\partial\Omega} g ds = 0,$$
$$\{g > 0\} = \Gamma_+,\ \{g < 0\} = \Gamma_-$$

A good θ for $n = 2$ and Ω simply connected

$\Delta\theta = 0$,

$\frac{\partial\theta}{\partial\nu} = g$ on $\partial\Omega$



$g : \partial\Omega \to \mathbb{R}$

$\int_{\partial\Omega} g ds = 0$,

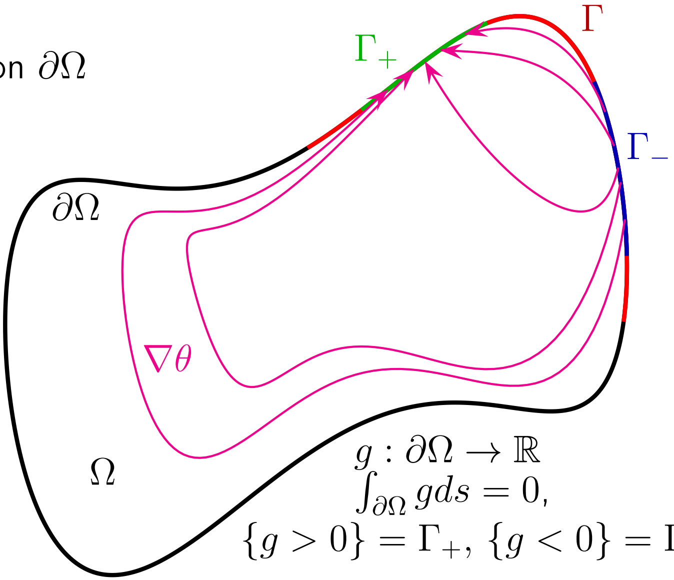
$\{g > 0\} = \Gamma_+$, $\{g < 0\} = \Gamma_-$

$\mathbb{R}^2$

A good θ for $n = 2$ and Ω simply connected

$\Delta\theta = 0,$

$\frac{\partial\theta}{\partial\nu} = g$ on $\partial\Omega$



$g : \partial\Omega \to \mathbb{R}$

$\int_{\partial\Omega} g ds = 0,$

$\{g > 0\} = \Gamma_+$, $\{g < 0\} = \Gamma_-$

$\mathbb{R}^2$

From local controllability to global controllability

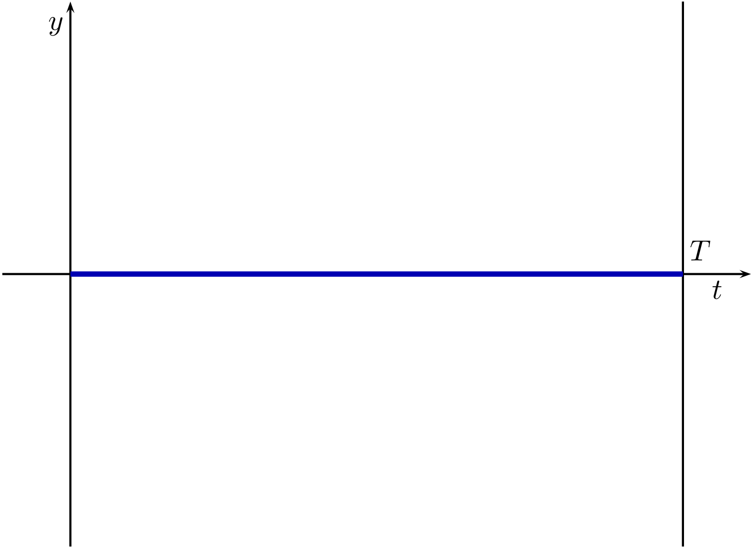
A simple scaling argument: if $(y,p) : [0,1]\times\overline{\Omega} :\to \mathbb{R}^n\times\mathbb{R}$ is a trajectory of our control system, then, for every $\varepsilon>0$, $(y^\varepsilon, p^\varepsilon) : [0,\varepsilon]\times\overline{\Omega}\to\mathbb{R}^n\times\mathbb{R}$ defined by

(1) $$y^\varepsilon(t,x) := \frac{1}{\varepsilon} y\left(\frac{t}{\varepsilon},x\right),\ p^\varepsilon(t,x) := \frac{1}{\varepsilon^2} p\left(\frac{t}{\varepsilon},x\right)$$

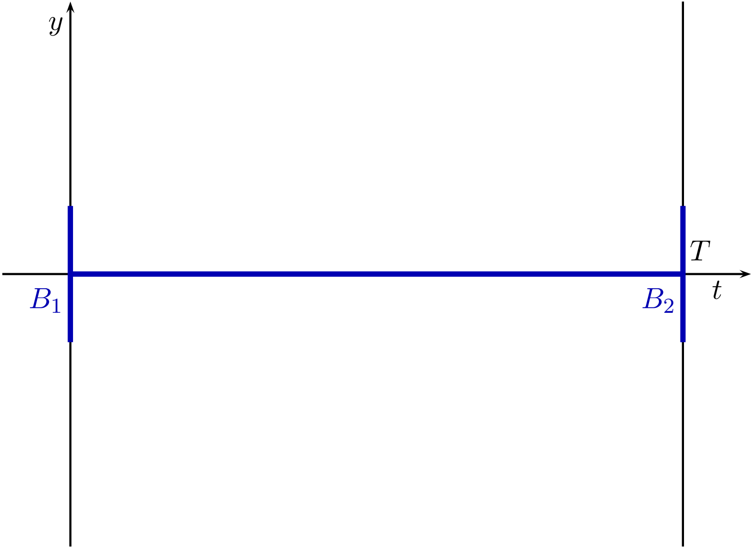
is also a trajectory of our control system.

Scaling

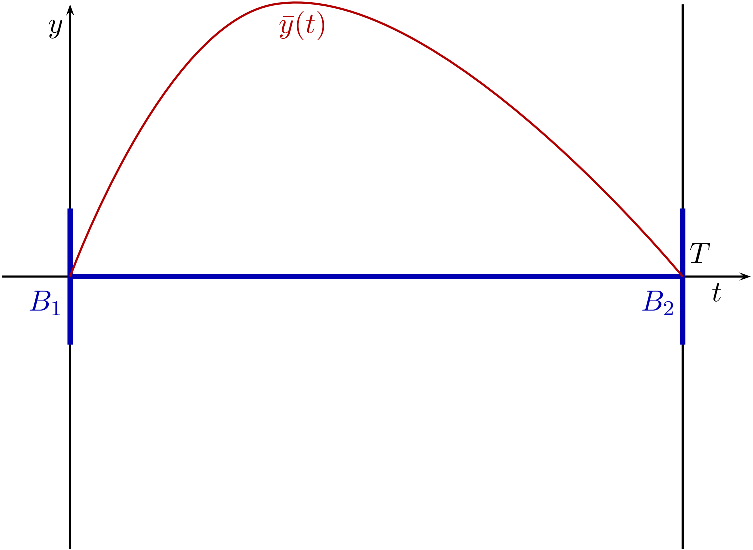
Scaling



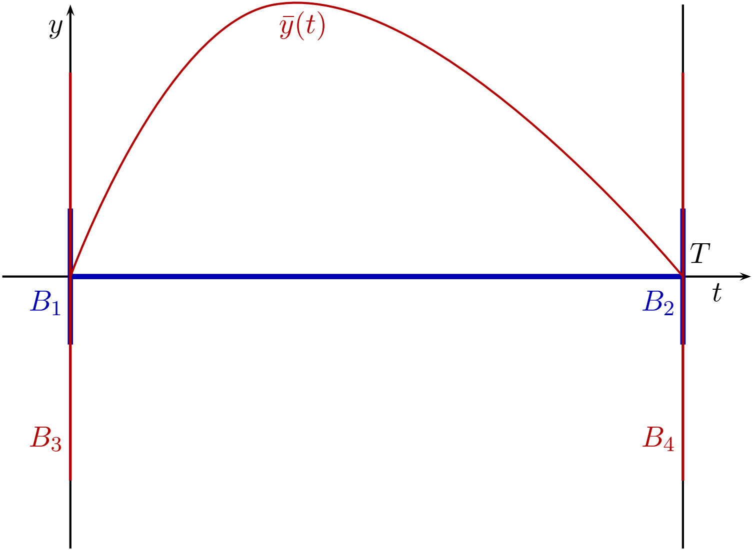
Scaling



Scaling



Scaling



An example in finite dimensional

We consider the following control system

$$\dot{y} = F(y) + Bu(t), \tag{1}$$

where the state is $y \in \mathbb{R}^n$, the control is $u \in \mathbb{R}^m$, B is a $n \times m$ matrix and $F \in C^1(\mathbb{R}^n; \mathbb{R}^n)$ is quadratic: $F(\lambda y) = \lambda^2 F(y)$, $\forall \lambda \in [0, +\infty)$, $\forall y \in \mathbb{R}^n$.
We assume that there exists a trajectory
$(\bar{y}, \bar{u}) \in C^0([0, T_0]; \mathbb{R}^n) \times L^\infty((0, T_0); \mathbb{R}^m)$ of the control system (1) such that the linearized control system around $(\bar{y}, \bar{u})$ is controllable and such that $\bar{y}(0) = \bar{y}(T_0) = 0$.

Remark

One has $F(0) = 0$. Hence $(0, 0)$ is an equilibrium of the control system (1). The linearized control system around this equilibrium is $\dot{y} = Bu$, which is not controllable if (and only if) B is not onto.

Let A be a $n \times n$ matrix and let us consider the following control system

$$\dot{y} = Ay + F(y) + Bu(t), \tag{1}$$

where the state is $y \in \mathbb{R}^n$, the control is $u \in \mathbb{R}^m$. For the application to incompressible fluids, (1) is the Euler control system and (1) is the Navier-Stokes control system.
One has the following theorem.

Theorem

Under the above assumptions, the control system (1) is globally controllable in arbitrary time: For every $T > 0$, for every $y^0 \in \mathbb{R}^n$ and for every $y^1 \in \mathbb{R}^n$, there exists $u \in L^\infty((0,T);\mathbb{R}^m)$ such that

$$\left(\dot{y} = f(y, u(t)),\, y(0) = y^0\right) \Rightarrow \left(y(T) = y^1\right).$$

Proof of the controllability theorem

Let $y^0 \in \mathbb{R}^n$ and $y^1 \in \mathbb{R}^n$. Let

$$\begin{array}{rccc} G: & \mathbb{R} \times L^\infty((0,T_0);\mathbb{R}^m) & \rightarrow & \mathbb{R}^n \\ & (\varepsilon, \tilde{u}) & \mapsto & \tilde{y}(T_0) - \varepsilon y^1 \end{array}$$

where $\tilde{y} : [0,T_0] \rightarrow \mathbb{R}^n$ is the solution of

(2) $$\dot{\tilde{y}} = F(\tilde{y}) + \varepsilon A \tilde{y} + B\tilde{u}(t),\ \tilde{y}(0) = \varepsilon y^0.$$

The map G is of class C^1 in a neighborhood of $(0,\bar{u})$. One has $G(0,\bar{u}) = 0$. Moreover $G'_{\tilde{u}}(0,\bar{u})v = y(T_0)$ where $y : [0,T_0] \rightarrow \mathbb{R}^n$ is the solution of

(3) $$\dot{y} = F'(\bar{y})y + Bv,\ y(0) = 0.$$

Hence $G'_{\tilde{u}}(0,\bar{u})$ is onto. Therefore, using the implicit function theorem, there exist $\varepsilon_0 > 0$ and a C^1-map $\varepsilon \in (-\varepsilon_0, \varepsilon_0) \mapsto \tilde{u}^\varepsilon \in L^\infty((0,T_0);\mathbb{R}^m)$ such that

(4) $$G(\varepsilon, \tilde{u}^\varepsilon) = 0,\ \forall \varepsilon \in (-\varepsilon_0, \varepsilon_0),$$

(5) $$\tilde{u}^0 = \bar{u}.$$

Let $\tilde{y}^\varepsilon : [0, T_0] \to \mathbb{R}^n$ be the solution of the Cauchy problem $\dot{\tilde{y}}^\varepsilon = F(\tilde{y}^\varepsilon) + \varepsilon A \tilde{y}^\varepsilon + B\tilde{u}^\varepsilon(t)$, $\tilde{y}^\varepsilon(0) = \varepsilon y^0$. Then $\tilde{y}^\varepsilon(T_0) = \varepsilon y^1$. Let $y : [0, \varepsilon T_0] \to \mathbb{R}^n$ and $u : [0, \varepsilon T_0] \to \mathbb{R}^m$ be defined by

$$y(t) := \frac{1}{\varepsilon}\tilde{y}^\varepsilon\left(\frac{t}{\varepsilon}\right), \, u(t) := \frac{1}{\varepsilon^2}\tilde{u}^\varepsilon\left(\frac{t}{\varepsilon}\right).$$

Then $\dot{y} = F(y) + Ay + Bu$, $y(0) = y^0$ and $y(\varepsilon T_0) = y^1$. This concludes the proof of the controllability theorem if T is small enough. If T is not small, it suffices, with $\varepsilon > 0$ small enough, to go from y^0 to 0 during the interval of time $[0, \varepsilon]$, stay at 0 during the interval of time $[\varepsilon, T - \varepsilon]$ and finally go from 0 to y^1 during the interval of time $[T - \varepsilon, T]$.

A drawback of this strategy

However this strategy has a serious drawback in the case of partial differential equations if "Ay requires more derivatives on y that $F(y)$". For example it seems difficult to deduce from the controllability of

(1) $$y_t + y_x = 0,\ y(t,0) = u(t),\ x \in (0,L),$$

in time $T > L$ the (null) controllability of

(2) $$y_t + y_x - \varepsilon y_{xx} = 0,\ y(t,0) = u(t),\ y(t,L) = v(t),\ x \in (0,L),$$

in time $T > L$ if $\varepsilon > 0$ is small enough. So, let us propose a slightly different strategy (requiring stronger assumptions).

A slightly different strategy

Let us, moreover, assume that the control system

$$\dot{y} = Ay + F(y) + Bu \tag{1}$$

where the state is $y \in \mathbb{R}^n$ and the control is $u \in \mathbb{R}^m$ is locally controllable in small time. Then one can proceed in the following way in order to get the global null controllability in small time of $\dot{y} = Ay + F(y) + Bu$. We want to send y^0 to 0 to 0 in small time by using a suitable control u. Again we perform the following scaling

$$z(t) := \varepsilon y(\varepsilon t),\; w(t) := \varepsilon^2 u(\varepsilon t). \tag{2}$$

Then $\dot{y} = Ay + F(y) + Bu$ is equivalent to $\dot{z} = \varepsilon Az + F(z) + Bw$. We then look for z and v of the following form

$$z = \bar{y} + \varepsilon z^1 + \varepsilon^2 z^2 + \dots,\; w = \bar{u} + \varepsilon v^1 + \varepsilon^2 v^2 + \dots \tag{3}$$

Then, identifying the orders in ε^p, $p \in \{0,1\}$ in $\dot{z} = \varepsilon z + F(z) + Bw$ one gets

$$\dot{\bar{y}} = F(\bar{y}) + B\bar{u}, \tag{1}$$

$$\dot{z}^1 = A\bar{y} + \frac{\partial F}{\partial y}(\bar{y},\bar{u})z^1 + \frac{\partial F}{\partial u}(\bar{y},\bar{u})w^1. \tag{2}$$

Note that, from our assumption on $(\bar{y},\bar{u})$, (1) holds. For the initial data, we have

$$\bar{z}(0) = 0,\ z^1(0) = y^0. \tag{3}$$

From (1) and the properties of $(\bar{y},\bar{u})$, one has $\bar{y}(T_0) = 0$. From our assumption of controllability of the linearized control around $(\bar{y},\bar{u})$ one gets the existence of v^1 such that $z^1(T_0) = 0$. So, with this w^1, $z(T_0)$ is of order ε^2. Going back to the y variable one gets that $y(\varepsilon T_0)$ is of order ε. Then using the local controllability in small time of $\dot{y} = Ay + F(y) + Bu$, one gets that, for every $\tau > 0$, we can find a control allowing us to go for the control system $\dot{y} = Ay + F(y) + Bu$ from $y(\varepsilon T_0)$ to 0 during the interval of time $[\varepsilon T_0, \varepsilon T_0 + \tau]$.

This gives again the global null controllability in small time of

$$\dot{y} = Ay + F(y) + Bu. \tag{1}$$

It requires an extra property, namely, the local null controllability in small time of (1), but it avoids the use of the inverse mapping theorem which is a serious problem in the pde framework if "Ay requires more derivatives on y that $F(y)$".

Morality

The "morality" behind these strategies is that in some cases the quadratic term $F(y)$ is the leading term compared to the linear term Ay for the global controllability: Ay is just an annoying perturbations (which can however be used when we are close enough to 0).
Let us try to apply this method to the global null controllability of the Navier-Stokes equation. As one can see by looking at the proof of the controllability theorem, this method works requires to have a (good) convergence of the solution of the Navier-Stokes equations to the solution of the Euler equations when the viscosity tends to 0. This is the case on manifolds without boundary, which, in our situation, corresponds to the case where the control is on the full boundary of Ω: $\Gamma := \partial\Omega$ (or in the case of interior control on a manifold without boundary). This is why we (JMC and A. Fursikov (1996)) are able to prove the global controllability of the Navier-Stokes only if the control is on the full boundary. When the control is only on part of the boundary, the global controllability is a challenging open problem.

Let us recall that, for manifolds with boundary, this convergence is not known even in dimension $n = 2$ if there is no control. More precisely, let us assume that Ω is of class C^∞, that $n = 2$ and that $\varphi \in C_0^\infty(\Omega; \mathbb{R}^2)$ is such that div $\varphi = 0$. Let $T > 0$. Let $y \in C^\infty([0,T] \times \overline{\Omega}; \mathbb{R}^2)$ and $p \in C^\infty([0,T] \times \overline{\Omega})$ be the solution to the Euler equations

$$(E) \begin{cases} y_t + (y \cdot \nabla)y + \nabla p = 0, \text{ div } y = 0, \text{ in } (0,T) \times \Omega, \\ y \cdot \nu = 0 \text{ on } [0,T] \times \partial\Omega, \\ y(0,\cdot) = \varphi \text{ on } \overline{\Omega}. \end{cases}$$

Let $\varepsilon \in (0,1]$. Let $y^\varepsilon \in C^\infty([0,T] \times \overline{\Omega}; \mathbb{R}^2)$ and $p^\varepsilon \in C^\infty([0,T] \times \overline{\Omega})$ be the solution to the Navier-Stokes equations

$$(NS) \begin{cases} y_t^\varepsilon - \varepsilon \Delta y^\varepsilon + (y^\varepsilon \cdot \nabla)y^\varepsilon + \nabla p^\varepsilon = 0, \text{ div } y^\varepsilon = 0, \text{ in } (0,T) \times \Omega, \\ y^\varepsilon = 0 \text{ on } [0,T] \times \partial\Omega, \\ y(0,\cdot) = \varphi \text{ on } \overline{\Omega}. \end{cases}$$

One knows that there exists $C > 0$ such that

(1) $$|y^\varepsilon|_{C^0([0,T];L^2(\Omega;\mathbb{R}^2))} \leqslant C, \ \forall \varepsilon \in (0,1].$$

One has the following challenging open problems.

Open problems (Convergence of Navier-Stokes to Euler as the viscosity tends to 0)

(i) *Does y^ε converge weakly to y in $L^2((0,T)\times\Omega;\mathbb{R}^2)$ as $\varepsilon\to 0^+$?*

(ii) *Let K be a compact subset of Ω and m be a positive integer. Does $y^\varepsilon_{|[0,T]\times K}$ converge to $y_{|[0,T]\times K}$ in $C^m([0,T]\times K;\mathbb{R}^2)$ as $\varepsilon\to 0^+$? (Of course, due to the difference of boundary conditions between the Euler equations and the Navier-Stokes equations, one does not have a positive answer to this last question if $K=\overline{\Omega}$.)*

The Navier Stokes equations with the Navier slip boundary condition

Let Ω be a smooth bounded non empty open subset of $\mathbb{R}^n$, $n \in \{2, 3\}$. We are interested in the Navier-Stokes equations

$$(1) \qquad \begin{cases} y_t - \Delta y + (y \cdot \nabla)\, y + \nabla p = 0,\ t \in [0, T], x \in \overline{\Omega}, \\ \text{div } y = 0,\ t \in [0, T], x \in \overline{\Omega}, \end{cases}$$

where, at time $t \in [0, T]$ and at the position $x \in \overline{\Omega}$, $y(t, x) \in \mathbb{R}^n$ is the velocity of the viscous incompressible fluid. We assume that we are able to prescribe y on a non empty open subset Γ of $\partial\Omega$.

The Navier slip boundary condition

The Navier slip boundary conditions are

$$(1) \qquad y \cdot \nu = 0 \quad \text{and} \quad [D(y)\nu + Ay]_{\tan} = 0 \text{ on } \partial\Omega \setminus \Gamma.$$

For a vector field f, we introduce $[f]_{\tan}$ its tangential part and $D(f)$ the rate of strain tensor (or shear stress) which are defined by:

$$(2) \qquad [f]_{\tan} := f - (f \cdot \nu)\nu, \; D_{ij}(f) := \frac{1}{2}\left(f^j_{x_i} + f^i_{x_j}\right).$$

Eventually, in (1), A is a smooth matrix valued function on $\partial\Omega$, describing the friction near the boundary. This is a generalization of the usual condition involving a single scalar parameter $\alpha \geq 0$ (i.e. $A = \alpha I_d$). For flat boundaries, such a scalar coefficient measures the amount of friction. When $\alpha = 0$ and the boundary is flat, the fluid slips along the boundary without friction and there is no boundary layers. When $\alpha \to +\infty$, the friction is so intense that the fluid is almost at rest near the boundary; condition (1) converges to the Dirichlet condition.

The controllability problem of the Navier control system

The question of small time global exact null controllability asks whether, for any $T > 0$ and any initial data y^0 (in some appropriate space), there exists a trajectory y defined on $[0, T] \times \Omega$, which is a solution to the Navier control system

$$(1) \qquad \begin{cases} y_t + (y \cdot \nabla)\, y - \Delta y + \nabla p = 0 \text{ in } (0,T) \times \Omega \\ \ \text{div } y = 0, \\ y \cdot \nu = 0 \text{ and } \quad [D(y)\nu + Ay]_{\tan} = 0 \text{ on } (0,T) \times (\partial\Omega \setminus \Gamma), \end{cases}$$

satisfying $y(0, \cdot) = y^0$ and $y(T, \cdot) = 0$. In this formulation, we see system (1) as an under-determined system. The controls used are the (implicit) boundary conditions on Γ and can be recovered from the constructed trajectory y itself.

Main result

Theorem (JMC, F. Marbach and F. Sueur (2020))

Assume that Γ is an open subset of $\partial\Omega$ which meets every connected component of $\partial\Omega$. Let $T > 0$ and $y^0 \in L^2(\Omega)$ be such that div $y^0 = 0$ and $y^0 \cdot \nu = 0$ on $\partial\Omega \setminus \Gamma$. Then there exists a solution of the Navier control system such that $y(0, \cdot) = y^0$ and $y(T, \cdot) = 0$.

Prior result: Global approximate controllability in $W^{-1,\infty}(\Omega)$ in small time, with a better convergence on compact subsets of Ω (JMC 1996). However $W^{-1,\infty}(\Omega)$ is not enough to get the null controllability with a local controllability result.

Open problems

1. *Can one replace "Γ meets every connected component of $\partial\Omega$" by "Γ is nonempty and Ω is connected"?*
2. *In dimension 2 our solutions are strong if the initial data is smooth (and satisfies the Navier slip boundary condition). We do not know if this property holds in dimension 3.*
3. *A main remaining challenging open problem: The case of the no-slip condition (problem raised by J.-L. Lions in the late 80's).*

Key ingredients for our global controllability result for the Navier-Stokes equations with the Navier slip boundary condition

There are five main ingredients

1. The return method together with the idea to consider by scaling the Navier-Stokes as some kind of perturbation of the Euler equation (JMC (1992, 1996)),
2. The controllability of the Euler equation (JMC (1996), 0. Glass (2002)),
3. The description of the evolution of the boundary layer due to D. Iftimie and F. Sueur (2011),
4. The dissipation method due to F. Marbach (2014),
5. The local null controllability result due to S. Guerrero (2006) (for the no-slip boundary condition: O. Imanuvilov (2001) and E. Fernández-Cara, S. Guerrero, O. Imanuvilov and J.-P. Puel (2004)).

Preparation of the boundary layer profile

When there is no more control (let say for $T \geq T_0$) the boundary layer equation reduces to the following heat equations on the half line $\xi \geq 0$ (where the slow variable $x \in \Omega$, x close to $\partial\Omega$ plays the role of a parameter):

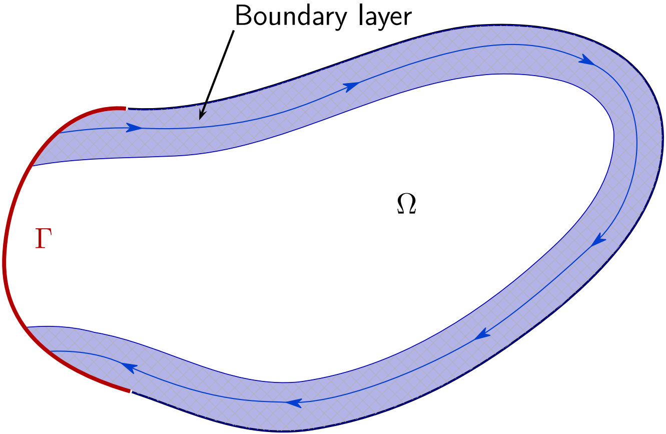
$$(1) \qquad \begin{cases} v_t - v_{\xi\xi} = 0, & (t, \xi, x) \in [T_0, +\infty) \times \mathbb{R}_+ \times \Omega, \\ v_\xi(t, x, 0) = 0 & (t, x) \in [T_0, +\infty) \times \Omega. \end{cases}$$

There is a natural dissipation on $[T_0, +\infty)$. Unfortunately this dissipation is not good enough for our purpose. However this dissipation turns out to be good enough if if the function v at time T_0 satisfies the following moment properties holds for $x \in \Omega$, x close to $\partial\Omega$,

$$(2) \qquad \int_0^{+\infty} \xi^k v(T_0, x, \xi) d\xi = 0, \ \forall k \in \{0, 1, 2, 3\}.$$

Property (2) can be obtained by using controllability properties of the boundary layer equation during the interval of time $[0, T_0]$ (even if this controllability is not sufficient to get $v(T_0, \cdot, \cdot) = 0$ since $\xi \in [0, +\infty)$).

Preparation of the boundary layer profile

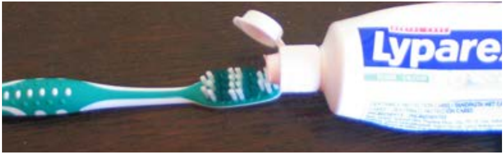


It is not possible to control the boundary layer. However we have a good enough control on it: we can modify so that it then dissipates quickly.

Main difficulty for the return method: Return to the initial state

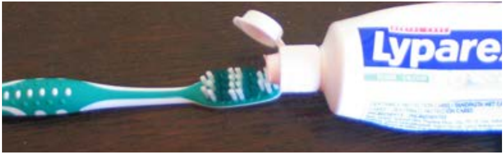
Main difficulty for the return method: Return to the initial state

Initial state



Main difficulty for the return method: Return to the initial state

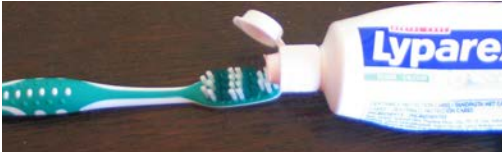
Initial state



Intermediate state

Main difficulty for the return method: Return to the initial state

Initial state



Intermediate state

It is difficult to return to the initial state.
"Once the toothpaste is out of the tube, it's going to be very hard to get it back in." H.R. Haldeman, 1973.

Toy model

$$\mathcal{T}\left\{\begin{array}{l}\mathcal{T}_1\left\{\ \dot{y}_1 = y_2,\ \dot{y}_2 = -y_1 + u,\right.\\ \mathcal{T}_2\left\{\ \dot{y}_3 = y_4,\ \dot{y}_4 = -y_3 + 2y_1y_2,\right.\end{array}\right.$$

where the state is $y = (y_1, y_2, y_3, y_4) \in \mathbb{R}^4$ and the control is $u \in \mathbb{R}$. The linearized control system of $\mathcal{T}$ around $(y_e, u_e) := (0, 0)$ is

(1) $$\dot{y}_1 = y_2,\ \dot{y}_2 = -y_1 + u,\ \dot{y}_3 = y_4,\ \dot{y}_4 = -y_3,$$

which is not controllable.

Controllability of the toy model

If $y(0) = 0$,

(1) $$y_3(T) = \int_0^T y_1^2(t)\cos(T-t)dt,$$

(2) $$y_4(T) = y_1^2(T) - \int_0^T y_1^2(t)\sin(T-t)dt.$$

Hence $\mathcal{T}$ is not controllable in time $T \leqslant \pi$. Using explicit computations one can show that $\mathcal{T}$ is (locally) controllable in time $T > \pi$.

Remark

For linear systems in finite dimension, the controllability in large time implies the controllability in small time. This is no longer for linear PDE. This is also no longer true for nonlinear systems in finite dimension.

How to recover the large-time local controllability of $\mathcal{T}$

The first point is at least to find a trajectory such that the linearized control system around it is controllable. We try the simplest possible trajectories, namely equilibrium points. Let $\gamma \in \mathbb{R}$ and define

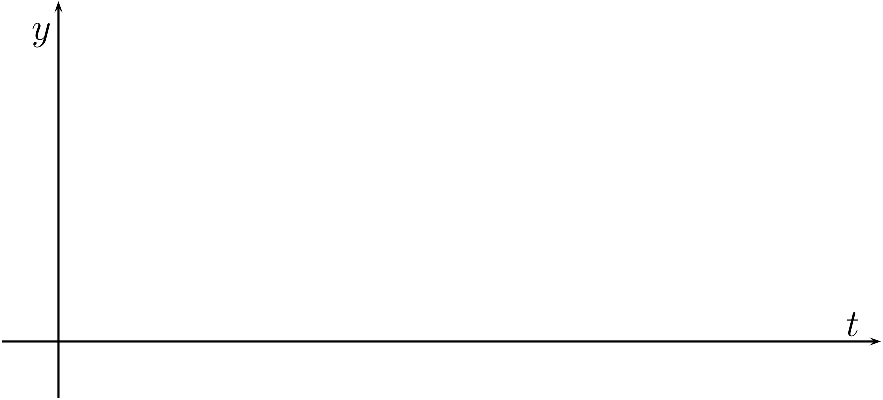
(1) $$((y_1^\gamma, y_2^\gamma, y_3^\gamma, y_4^\gamma), u^\gamma) := ((\gamma, 0, 0, 0), \gamma).$$

Then $((y_1^\gamma, y_2^\gamma, y_3^\gamma, y_4^\gamma), u^\gamma)$ is an equilibrium of $\mathcal{T}$. The linearized control system of $\mathcal{T}$ at this equilibrium is

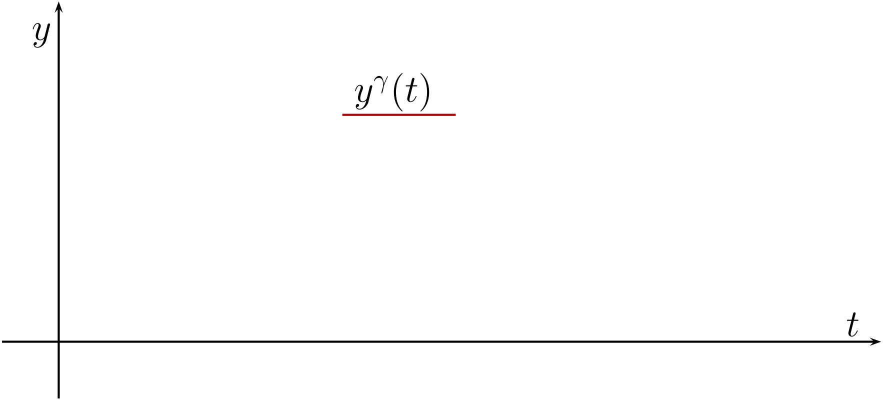
(2) $$\dot{y}_1 = y_2,\ \dot{y}_2 = -y_1 + u,\ \dot{y}_3 = y_4,\ \dot{y}_4 = -y_3 + 2\gamma y_2,$$

which is controllable if (and only if) $\gamma \neq 0$.

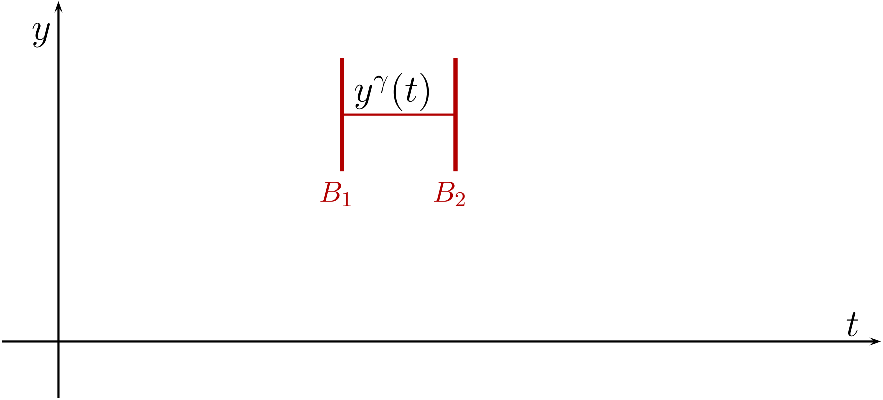
How to recover the large-time local controllability of $\mathcal{T}$



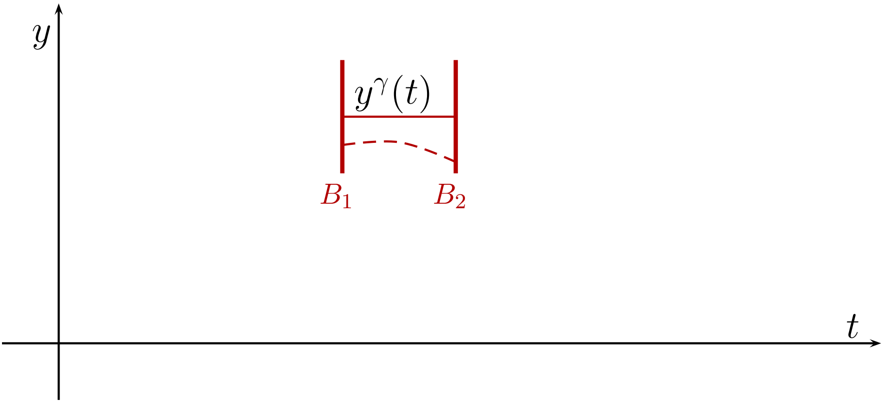
How to recover the large-time local controllability of $\mathcal{T}$



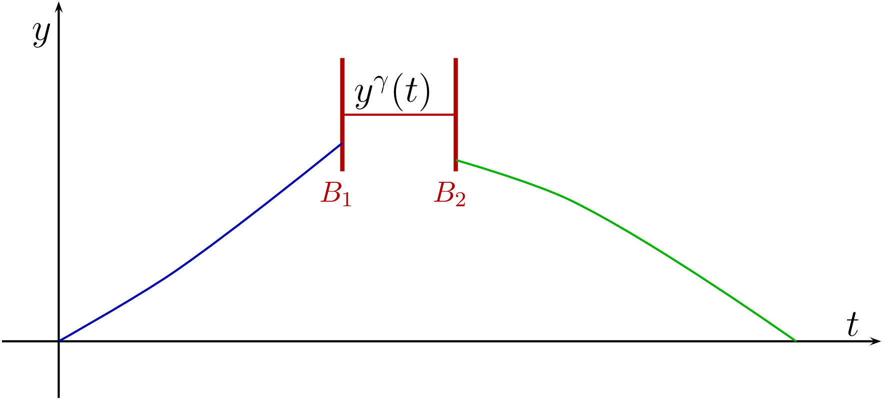
How to recover the large-time local controllability of $\mathcal{T}$



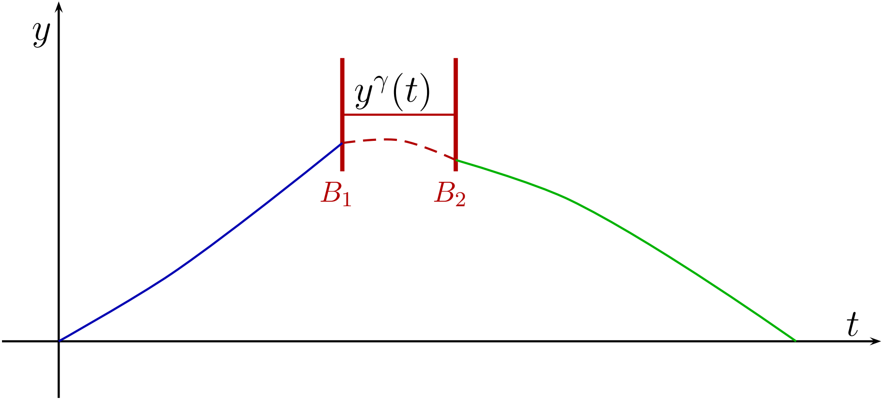
How to recover the large-time local controllability of $\mathcal{T}$



How to recover the large-time local controllability of $\mathcal{T}$



How to recover the large-time local controllability of $\mathcal{T}$



Construction of the blue trajectory

One uses **quasi-static deformations**. Let $g \in C^2([0,1];\mathbb{R})$ be such that

$$g(0)=0,\ g(1)=1.$$

Let $\tilde{u} : [0,1/\varepsilon] \to \mathbb{R}$ be defined by

$$\tilde{u}(t) := \gamma g(\varepsilon t),\ t \in [0,1/\varepsilon].$$

Let $\tilde{y} := (\tilde{y}_1, \tilde{y}_2, \tilde{y}_3, \tilde{y}_4) : [0,1/\varepsilon] \to \mathbb{R}^4$ be defined by requiring

$$\dot{\tilde{y}}_1 = \tilde{y}_2,\ \dot{\tilde{y}}_2 = -\tilde{y}_1 + \tilde{u},\ \dot{\tilde{y}}_3 = \tilde{y}_4,\ \dot{\tilde{y}}_4 = -\tilde{y}_3 + 2\tilde{y}_1\tilde{y}_2, \tag{1}$$

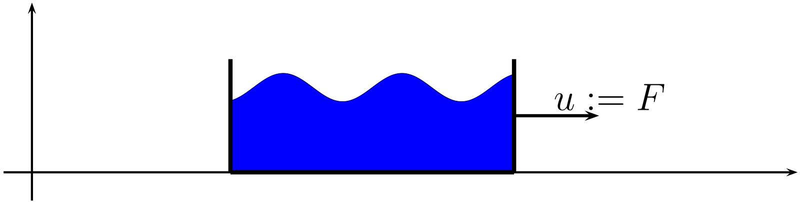
$$\tilde{y}(0) = 0. \tag{2}$$

One easily checks that

$$\tilde{y}(1/\varepsilon) \to (\gamma, 0, 0, 0) \text{ as } \varepsilon \to 0.$$

Quasi-static deformations: Applications

• The quasi-static deformation method has been introduced in JMC (2002) to prove the controllability of the water-tank control system.



• The quasi-static deformation method has also been used for the control of Schrödinger equation: K. Beauchard (2005), K. Beauchard and JMC (2006).