

Colloquium du CERMICS



**The Cauchy problem treated
as a convex optimization problem**

Yann Brenier (CNRS, DMA-ENS, PSL)

5 mars 2018

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en association avec le projet CNRS-INRIA "MOKAPLAN"

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An important class of evolution PDEs

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$$\partial_t U + \nabla \cdot (F(U)) = 0, \quad U = U(t, x) \in \mathcal{W} \subset \mathbb{R}^m,$$

$t > 0, x \in \mathbb{R}^d, \nabla = \left(\frac{\partial}{\partial x_i} \right)_{i=1}^d$, such that $\sum_{\beta=1}^m \partial_\beta \mathcal{E}(W) \partial_\alpha F^{i\beta}(W) = \partial_\alpha Q^i(W), \forall W \in \mathcal{W}$,

for some smooth pair $(\mathcal{E}, Q) : \mathcal{W} \rightarrow \mathbb{R}^{1+d}$ (with \mathcal{W} open convex and \mathcal{E} , called "entropy", strictly convex), which implies $\partial_t(\mathcal{E}(U)) + \nabla \cdot (Q(U)) = 0$, for all smooth solutions U .

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A famous example: the equations of an isothermal gas (Euler, 1755)

$$\partial_t \rho + \nabla \cdot \mathbf{q} = 0, \quad \partial_t \mathbf{q} + \nabla \cdot \left(\frac{\mathbf{q} \otimes \mathbf{q}}{\rho} \right) + \nabla \rho = 0, \quad \mathcal{E} = \frac{|\mathbf{q}|^2}{2\rho} + \rho \log \rho, \quad \rho > 0, \quad \mathbf{q} \in \mathbb{R}^d.$$

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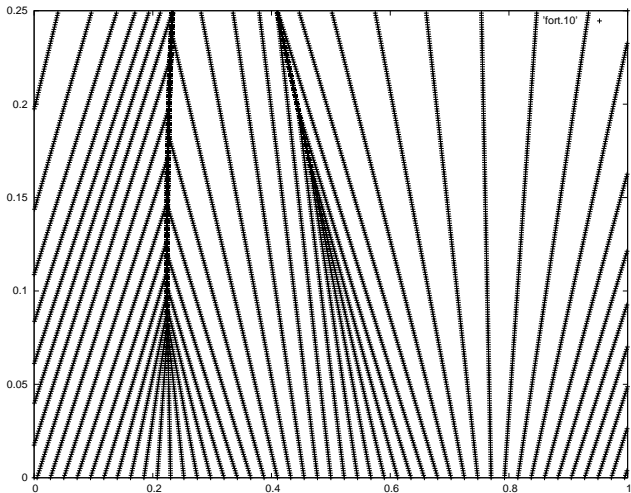
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Typically, these systems are locally well-posed, with generic formation of shock waves.



Inviscid Burgers equation : $\partial_t u + \partial_x(u^2/2) = 0$, $u = u(t, x)$, $x \in \mathbb{R}/\mathbb{Z}$, $t \geq 0$.
 Formation of two shock waves. (Vertical axis: $t \in [0, 1/4]$, horizontal axis: $x \in \mathbb{T}$.)

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The problem is not trivial since there may be many weak solutions starting from U_0 which are not entropy-preserving (by "convex integration" à la De Lellis-Székelyhidi).

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N.B. The supremum in A exactly encodes that U is a weak solution with initial condition U_0 , all test functions A acting like Lagrange multipliers.

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leads to a *concave* maximization problem in A , namely

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$$= \sup_A \int_{[0, T] \times \mathbb{T}^d} -K(\partial_t A, \nabla A) - \int_{\mathbb{T}^d} A(0, \cdot) \cdot U_0$$

$$K(E, B) = \sup_{V \in \mathcal{W} \subset \mathbb{R}^m} E \cdot V + B \cdot F(V) - \mathcal{E}(V), \quad (E, B) \in \mathbb{R}^m \times \mathbb{R}^{m \times d}.$$

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Notice that K is automatically convex.

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For instance, if K were just a square, we would get

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Answer: in our construction, K is *convex degenerate!*

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(*) more precisely if, $\forall t, x, V \in \mathcal{W}$, $\mathcal{E}''(V) - (T - t)F''(V) \cdot \nabla(\mathcal{E}'(U(t, x))) > 0$, which requires, in particular, $\mathcal{E}''(V) - TF''(V) \cdot \nabla(\mathcal{E}'(U_0(x))) > 0$.

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...But what about shocks and large T ???

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Then, the maximization problem in A simply reads

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with $A = A(t, x) \in \mathbb{R}$ subject to $A(T, \cdot) = 0$, $\partial_x A \leq 1$.

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Introducing $\rho = 1 - \partial_x A \geq 0$, $q = \partial_t A \in \mathbb{R}$, this problem is equivalent to

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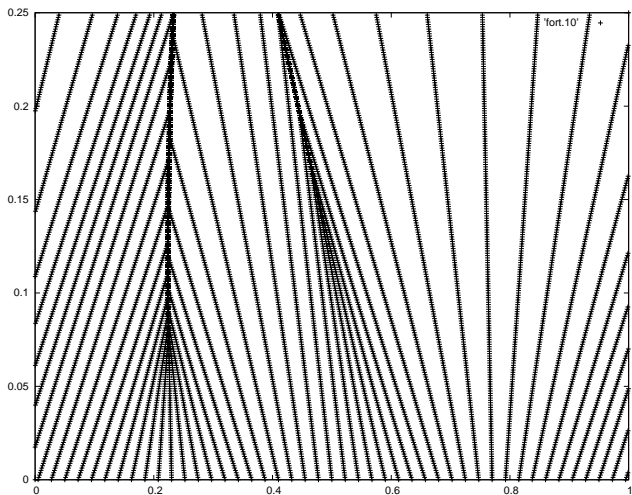
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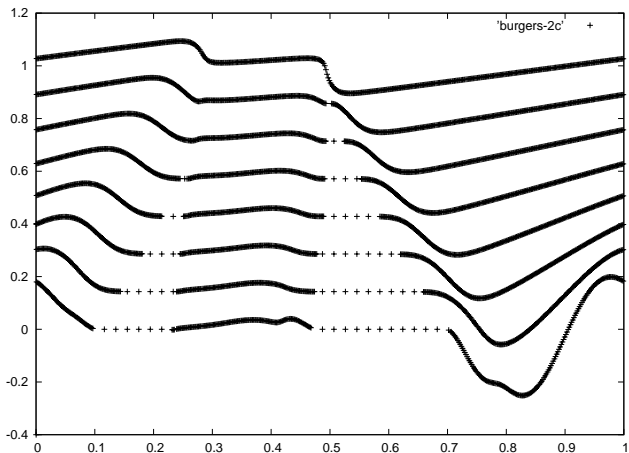
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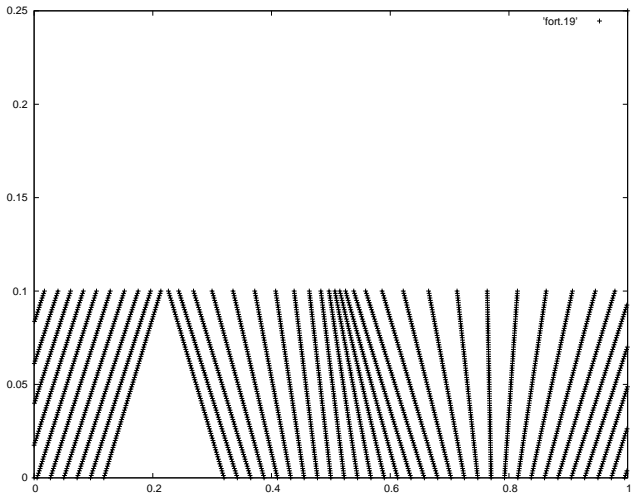
i.e. the "ballistic" version (à la Ghoussoub) of the **optimal transport problem** with quadratic cost (à la Benamou-B.), and, as well, an elementary "mean-field game".



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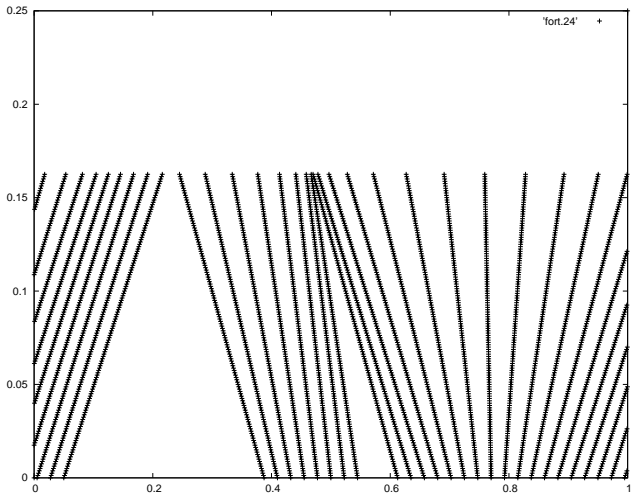
Numerics: 2 lines of code differ from a standard (Benamou-B.) OT solver!



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Recovery of the solution at time $T=0.1$ by convex optimization.

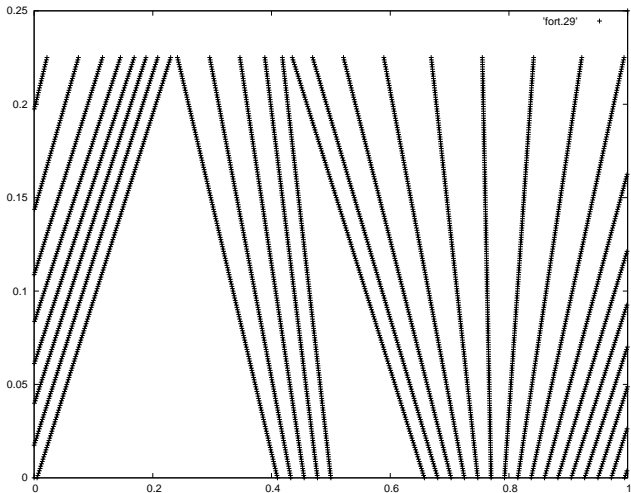
Observe the formation of a first vacuum zone as the first shock has formed.



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Recovery of the solution at time $T=0.16$ by convex optimization.

Observe the formation of a second vacuum zone as the second shock has formed.



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Recovery of the solution at time $T=0.225$ by convex optimization.

Observe the extension of the two vacuum zones.



Analogy with mountain climbing:
going from Everest to Lhotse without following the crest! (Credit to Th. Gallouët.)

Merci de votre attention!

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Pour plus de détails, voir Y.B. ArXiv Oct. 2017.